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ABSTRACT

This book, written by participants of a joint Japan-United States seminar, drawing upon years of classroom and research experience, examines how computation applies in basic and far-reaching ways, starting in early childhood and continuing throughout individuals' lives. The book presents both the Japanese and American views of computational alternatives--mental, written, with calculators, and through estimation--and their implications for school mathematics at all levels. The essays are categorized into the following sections: (1) "Similarities and Differences--Mental Computation in Japan and the United States," (2) "Similarities and Differences--Computational Estimation in Japan and the United States," and (3) "Similarities and Differences--Calculator Use in Japan and the United States." Contains 170 English and 49 Japanese references. (MKR)

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Computational Alternatives for the Twenty-first Century

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Cross-Cultural Perspectives from Japan and the United States

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Computational Alternatives for the Twenty-first Century

Cross-Cultural Perspectives from Japan and the United States

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PREFACE

Computation applies mathematics in basic and far-reaching ways, beginning in early childhood and continuing throughout our lives. Computation provides a valuable and useful connection between problem solving and mathematical concepts. Although the nature and form of computations vary, every culture in every country of the world uses computation. No one questions the value of computation, but many people are questioning what should be done to develop computation that is commensurate with current and future needs.

Cries for significant change are coming from different sources in various countries. For example, the *Curriculum and Evaluation Standards for School Mathematics* and the *Professional Standards for Teaching Mathematics* call for significant change in mathematics programs (National Council of Teachers of Mathematics 1989, 1991). In a similar spirit, the *Curriculum of Mathematics for the Elementary School* provides new directions for Japanese mathematics programs (Japanese Ministry of Education 1989a).

School mathematics programs preparing students for the next century must carefully examine the role of computational alternatives—mental, written, calculators, and estimation. More specifically, plans for providing a balance of computational alternatives need to be discussed, debated, and developed for all grade levels. In response to this situation and the calls for major change, a joint Japan–United States seminar, "Computation for the Twenty-first Century: Cross-Cultural Perspectives," was held in Honolulu, Hawaii, in 1990. The week-long seminar, organized by Nobuhiko Nohda and Robert Reys, presented an opportunity to examine the current state of computational alternatives within each country and explore needed direction for change.

Invitations were extended to mathematics educators who had made significant contributions in one or more aspects of computational alternatives within their respective countries. Each participant prepared a paper and made a presentation that focused on an issue related to computational alternatives at the elementary school level. The seminar provided keynote presentations, full-group discussions—including questions, clarification, and dialogue related to the presentations—and small-group discussions, which often focused on needed research. Overall, the seminar furnished valuable insights into what had happened, what is happening, and what is likely to happen in the near future in each country regarding computational alternatives.

On completion of the seminar, it was felt that the presentations and ideas expressed would be of interest to other mathematics educators, not only in Japan and the United States but in other countries as well.

This publication is the culmination of three years' effort to capture the central themes addressed during the seminar and mold them into this monograph.

The opening chapter discusses computational alternatives and their importance in school mathematics programs. It also profiles each of the subsequent chapters, which provide both historical and contemporary discussion of these alternatives and their implications for school mathematics. A Japanese view of each alternative is provided by Japanese mathematics educators, and an American view of the same alternative is provided by American mathematics educators.

Throughout the seminar, participants were asked to reflect on similarities and differences between the two countries' approach to each of the computational alternatives. The discussions and resulting written summaries were helpful in better understanding the position of each country. Although there was not always unanimous agreement—or sometimes even consensus—on what constituted similarities or differences, these summaries furnish a brief and useful distillation of the chief issues. They help characterize some of the cultural differences and similarities surrounding computational alternatives within Japan and the United States. The summaries have been strategically placed to precede each cluster of related chapters. It is expected that these summaries of similarities and differences will provide an advance organizer for some readers and a useful overview for others.

We think this monograph makes for interesting reading. We hope that it will stimulate some of the same excitement, interest, and fresh thinking about computational alternatives that we have experienced. Further, we hope that it will inspire more people to become involved with research related to computational alternatives and to implement more of what is already known into widespread use in school mathematics programs around the world.

Special thanks are owed to the University of Missouri and the Weldon Springs Research Fund, which supported the final development of this monograph and allowed it to become a reality. Additional thanks should be given to the National Science Foundation and the Japan Society for the Promotion of Science for their support of the original joint seminar, and also to the National Science Foundation for a long-term research leave to Japan, which allowed much of the critical development of this monograph to take place.

We also thank all the participants and observers involved in this joint seminar. We appreciate the contributions each of them made. Every author not only prepared her or his paper but often edited other papers. The authors worked diligently to produce papers of high quality, adjusted personal schedules to meet some challenging deadlines, and were

consistently cooperative in accepting critiques and rewriting to reflect countless editorial and reviewer suggestions. Without their expertise and willingness to contribute their time and energy, this publication would never have become a reality. In addition, we thank Bonnie Schappele, San Diego State University, and Frank Lester, Indiana University, for their help in the review process.

1

Computation and the Need for Change

Robert E. Reys
Nobuhiko Nohda

MUCH preparation is already under way for entering the twenty-first century. Education—which profoundly affects all aspects of our lives, including our livelihood and quality of living—is receiving great attention. The education community is carefully examining the needs of students who will be spending their lives coping with challenges and solving problems that we cannot anticipate or even contemplate. Although there is much speculation about what is needed in schools for the future, there is a consensus that education will assume an ever-greater importance in the future and that the knowledge base in science and mathematics will provide an essential stepping stone for an increasing number of career opportunities. All signs indicate that the need for a mathematically literate population not only will continue to grow but will actually accelerate rapidly. There is currently much activity in the mathematics education community designed to communicate a new vision for mathematics in schools that will help future mathematics programs better prepare students for a productive life in the next century.

A persistent theme of the current reform movement in school mathematics is the need to develop in students a deep understanding of mathematical ideas, an ability to communicate and reason mathematically, and the capability to use their knowledge to solve a wide range of problems (*Curriculum and Evaluation Standards for School Mathematics* [National Council of Teachers of Mathematics 1989]). A second persistent theme in the reform is the need to decrease the amount of emphasis on traditional paper-and-pencil arithmetic and develop a broader, more balanced approach to computation (*Everybody Counts: A Report to the Nation on the Future of Mathematics Education* [National Research Council 1989]; *Reshaping School Mathematics: A Philosophy and Framework for Curriculum* [Mathematical Sciences Education Board 1990]). For many people throughout the world, mathematics is synonymous with arithmetic and arithmetic with written computations. Constructing mathematics programs that reflect a balanced approach to computation is a challenge being addressed internationally (Cockcroft 1986; Japanese Ministry of Education 1989; Willis 1990).

The image teachers and many lay persons have about computation must be changed. A broad conception of computation is needed, and one insightful "global view" of computation was provided by Coburn (1989). His global view connects the computational alternatives of mental computation, written computation, estimation, and calculators. This model helps communicate a broad conception of computation.

A review of recent reports (National Research Council 1989, 1990; National Council of Teachers of Mathematics [NCTM] 1989; Japanese Ministry of Education 1989; Willis 1990) presents compelling arguments for helping students develop and use different computation procedures. For example, figure 1.1 illustrates a view of computation presented in the NCTM *Curriculum and Evaluation Standards* (p. 9).

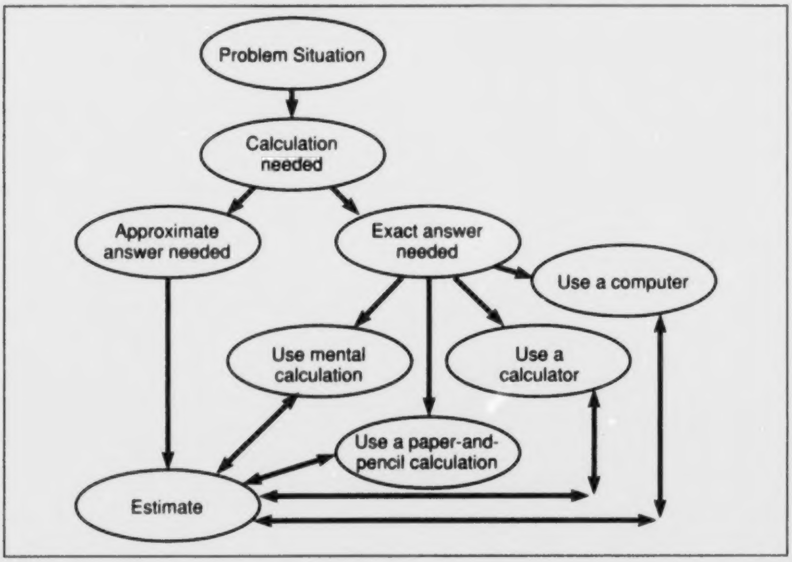
Problem Situation
Calculation needed
Approximate answer needed
Exact answer needed
Use a computer
Use mental calculation
Use a calculator
Use a paper-and-pencil calculation
Estimate

Fig. 1.1

It begins with a problem situation involving calculation and then highlights the series of decision-making stages that must be completed when doing computation. In every computational problem, one must make a decision about what type of result is needed and how that result will be produced. Figure 1.1 highlights an important goal of mathematics education, namely, to help people make wise choices among the computational alternatives, correctly apply those choices to produce a result, and use number sense to determine the reasonableness of their "answer" (result) for the particular problem being solved.

Other reports contain clear and direct comments calling for more attention to computational alternatives. Here are a few examples:

> [From *Reshaping School Mathematics*] Elementary school mathematics should use ... calculators. It should emphasize mental arithmetic, particularly for estimating.... At the same time there should be a sharp reduction in time devoted to teaching the traditional written methods of calculation for multidigit numbers, fractions, and decimals. (Mathematical Sciences Education Board 1990, p. 41)
>
> [From *Everybody Counts*] Children should use calculators throughout their school work, just as adults use calculators throughout their lives. More important, children must learn when to use them and when not to do so. They must learn from experience with calculators when to estimate and when to seek an exact answer; how to estimate answers to verify the plausibility of calculator results; and how to solve modest problems mentally when neither pencil nor calculator is convenient. (National Research Council 1989, p. 47)
>
> There are several merits for the use of calculators....We will be able to treat the principles and conceptions through realistic approach. Secondly, it will be possible to have understand [*sic*] more completely with a lot of numerical values. Thirdly, especially at elementary school level, learning on computation and learning on which operation should be applied will be separated. Shimada (1985) characterized the second one as a magnifier or a microscope, namely, a tool for discovery, providing rich data which cannot be obtained by naked eyes only, and the third one as glasses for short sight, namely, as an aid for computation, covering weak points in paper-pencil computational skills. (Nagasaki 1987a, p. 66)

The winds of change are blowing strongly across the mathematics curricula of Japan and the United States, but computation has a strong and well-developed root system (textbooks, instructional practices, and parental expectations) that makes significant change slow and often difficult.

Computation originated to fill a real-world problem-solving need. Its use has been documented in the earliest records of many different cultures. Through the years computational development was facilitated by the gradual evolution of a variety of counting and calculating devices (such as stones, the abacus, soroban, slide rule, and calculator) and algorithmic procedures.

It is safe to say that computation, more specifically written computation, has been the cornerstone on which elementary school mathematics programs have been built. Japan and the United States are culturally very different, yet historically computation not only has been an integral part of mathematics in both countries but has dominated their elementary school programs. Furthermore, school-based mathematics has tended to focus on exact computation and has stressed the development of written computation.

Years have been spent developing the mastery of computational skills with whole numbers, fractions, and decimals, and for the most part this development centered on establishing written algorithms to do tedious computations. These skills may have been needed in the past, but they hold no promise for the future, since adults now rarely do tedious written computations. Research of adult usage shows that more than 80 percent of all mathematical computations in daily life involve mental manipulation of numerical quantities rather than written computations. Ironically, research also shows that 70 to 90 percent of the time in elementary school mathematics instruction in the United States is directed toward written computation (Suydam and Weaver 1981). Similar research in Japan has reported that much time is devoted to teaching written computation (Itoh 1987).

During the last third of this century, hand calculators have become available. The widespread availability and easy access to calculators has provided another viable computational alternative when exact results are needed. Although calculators are easy to use, errors are also easily made, and this has placed a higher premium on developing computational estimation skills as one important means of determining if calculator-produced results are reasonable. Calculators have further challenged mathematics educators to examine what computations should be learned in school, but the eternal questions regarding how these computations should be learned continues to stir debate.

SOME DIFFICULT AND CHALLENGING QUESTIONS

As plans accelerate to develop school mathematics programs to prepare students for living in the twenty-first century, it is clear that the attention to computation will change. The previously cited reports testify that there is unanimous agreement that a better balance of computational alternatives—including mental computation, estimation, and calculator use—must be reflected in school mathematics programs. These reports call for a broadening of the treatment of computation, not necessarily a reduction of the time devoted to computation. This means the attention to computation must shift. Much less time should be devoted to written computation, and much more time should be given to developing mental computation, estimation, and calculator use. This plan sounds reasonable but is far from simple. The only thing certain is that the implementation of this plan would dramatically change elementary school mathematics programs.

Although computational alternatives will receive increased attention in this broadened conception of computation, no proposals for prototype curricula have been offered. Much debate and discussion of major issues is currently taking place, however. Among the pragmatic questions being raised and addressed are these:

- What is number sense?
- What is mental computation?
- What is estimation?
- What is written computation?
- How are mental and written computation intertwined?
- What is the notion of reasonableness?
- How is it developed?
- How is it related to number sense?
- What is the role of written algorithms in the twenty-first century? Should there be limits on the complexity of the written computations that students are expected to master?
- How should computational alternatives (mental computation, estimation, written algorithms, calculators) be developed? When should computational alternatives be introduced? Should strategies and techniques be self-developed by students, as advocated by constructivists? Or should strategies and techniques be taught directly by teachers?
- In what order should these alternatives be developed? What levels of performance can be achieved?
- How are wise choices of computational alternatives developed? Do students know when mental computation is appropriate? Do students know when an exact answer is needed or when an estimate is sufficient?
- How can calculators be used? Can calculators contribute to the development of mathematical thinking? How? What role does the calculator play as a tool? Where? How?
- How does the development of computational alternatives contribute to number sense?
- How can valid measures of the computational alternatives be obtained?
- How should tests be changed to accommodate the availability and use of calculators?
- Does the development of computational alternatives reinforce or enhance problem solving? How?

This list of questions is not exhaustive. However, these questions are among those being pondered, debated, and addressed by mathematics educators in countries throughout the world. At the very least, such questions represent a fertile area for research. Answers to them are needed before substantial progress can be made toward successfully implementing the array of computation alternatives. Moreover, each country has additional questions that require thoughtful self-examination. For example, in Japan the soroban has been an integral part of the culture for centuries. Research suggests that the soroban has been a powerful model for developing mental computation, so a natural question relates to the role the soroban should play in the future. How should the soroban and calculators be integrated into school mathematics programs? Certainly many significant questions need to be addressed by each country. However, there is also the need to rise above a nationalistic view and examine common issues that are of mutual concern. Here again research is needed, but of the cross-cultural variety.

ELEMENTARY SCHOOL MATHEMATICS PROGRAMS IN JAPAN AND THE UNITED STATES

Educational systems are complex and reflect unique cultural values of a country. Knowledge of the many facets of an educational system—such as, How are societies' expectations for schools identified? How are educational goals established? Schools organized? Curricula developed? Teachers prepared? Examinations constructed?—is essential to truly "understanding education" in a country. Lack of space makes it impossible to discuss these questions here. We do, however, suggest that interested readers examine several recent publications that furnish a contemporary perspective of the Japanese educational system (Stevenson 1991; Stevenson and Stigler 1992; Kumano 1991; Ministry of Education, Science, and Culture 1991).

Interest in establishing a more balanced approach to computation is growing in Japan and the United States. Serious attention is now being given to developing computational alternatives in both countries. However, it is recognized that the two countries differ greatly in the dynamics of change. For example, in Japan educational change is typically orderly and carefully planned. In the United States, the entire process of educational change is often chaotic—with public battles between ideologies, local implementation problems, and great diversity of practice. Technology, more specifically the calculator, provides a classic example of public battles and diversity of practice in action. On no single issue has debate been more heated and emotional than on the role of calculators in school mathematics.

Consider further how curriculum materials for elementary school mathematics programs are developed. On the one hand, the United

States has no national mathematics curriculum. Local school boards set policy within the limits imposed by their state departments of education. Thus, all curricular and instructional decisions are made by states, districts, and local schools. There are almost as many mathematics curricula as there are school districts. Although new textbooks in the United States are influenced by recommendations from respected professional groups and organizations, the curriculum materials produced by commercial companies are primarily market driven. Consequently, the main criterion for change is often "what will the teachers really buy?" Changes may be cosmetic and superficial rather than extensively and carefully integrated throughout the curriculum. On the other hand, Japan has a national mathematics curriculum that is established by the Ministry of Education. It furnishes guidelines for the development of new mathematics programs, such as the latest *Curriculum of Mathematics for the Elementary School* (1989a). Although there are six commercial textbook companies in Japan, each must be approved by the Ministry of Education before they are permitted to sell their programs. Once curricular changes are recommended by the Ministry of Education, the Japanese approach insures a rapid and consistent change in mathematics curricula.

An examination of the physical characteristics of the Japanese and American mathematics textbooks also shows sharp differences. Japanese textbooks, in contrast to the encyclopedic nature of United States textbooks, are brief guides to the teacher rather than comprehensive and detailed daily lessons with prescribed instructional methods. For example, the Japanese books are much smaller (typically 15 cm by 21 cm) and have fewer than half the number of pages of their American counterparts. Whereas much space is given to practice exercises in American textbooks, only a few exercises are typically included in Japanese student books. The majority of the pages in Japanese textbooks are devoted to concept and skill development. Furthermore, there are no end-of-chapter pages for review, reteaching, enrichment, and chapter tests like those typically found in American textbooks.

The grade level at which different mathematical topics are introduced and developed in each country varies. In general, the pace at which computation is introduced and established is much more rapid in Japan than in America. For example, in Japan, basic multiplication facts are introduced and students are expected to master them in second grade, whereas multiplication is not typically introduced until third grade in the United States and often not mastered until later. In Japan, nearly all computation has been established by the end of grade 5, whereas in the United States computation continues to be developed, practiced, and reviewed through junior high school.

Each country's approach to teaching mathematics is also shaped by its national character. In Japan, the standard nationwide curriculum influences not only what mathematics is taught but also the instructional methods. Observations of mathematics teaching in Japanese schools suggest a remarkable similarity in teaching style and procedures. In America, there is a wide variety of instructional approaches, ranging from force-feeding facts to encouraging children to construct their own mathematical understandings.

A PREVIEW OF THE UPCOMING DISCUSSION

Mental Computation

In Japan, mental computation has been viewed as important and has been consistently developed for centuries. The soroban (Japanese abacus) was introduced from China. As the soroban becomes internalized, its use in calculation has also aided the development of mental computation. In the next paper, Shibata traces the development of mental computation and discusses the evolution of the Japanese terms *anzan* and *hissan*, which are associated with mental and paper-and-pencil computation and yet are used differently than similar terms in the United States. The Japanese paper by Shigematsu, Iwasaki, and Koyama gives an insightful examination of curricular and instructional issues related to mental computation. Ironically, the Japanese express some concerns that the attention given to the soroban and mental computation in the latest school curriculum has been decreased.

Whereas mental computation has received consistent attention in Japan, the story in the United States is quite different. In the early part of the century, mental computation was emphasized. However, since the 1920s, the emphasis on mental computation has varied greatly. Mental computation is currently experiencing a resurgence of support in an effort to help students develop greater expertise in mental computation as well as provide an important prerequisite for estimation. A profile of some changes related to mental computation is given by Reys and Barger.

Computational Estimation

Computational estimation is a focal point of much attention in both countries. Computational estimation has received consistently strong support from professional groups and curricular committees in the United States. However, this support has been insignificantly reflected in school mathematics programs. It is not clear that schools or teachers contemplating curriculum change have ever addressed two critical questions, namely: (1) What is computational estimation? and (2) How should it be developed?

In the United States, computational estimation has not received consistent attention, and it has not been addressed in the context of all numbers or all operations. Rounding has been taught as a skill, and assumptions have been made that these rounding skills would facilitate the development of estimation. In reality, research suggests that rigid rules for rounding often interfere with natural estimation processes and stifle the development of computational estimation. Not only has computational estimation been given little emphasis (often misdirected) in elementary school, but it has received virtually no attention in secondary school. Unfortunately, the attention given to computational estimation is often isolated and limited to distinct lessons—lessons and ideas that are rarely integrated or reviewed in other computational activities. In later chapters, Trafton elaborates on critical instructional issues related to estimation, and Schoen addresses concerns and difficulties in assessing students' knowledge and understanding of estimation.

Historically, computational estimation has received very little attention in Japan, but the latest curricular recommendations identify estimation as "an important ability in an information society" and a process that should be specifically taught within elementary school mathematics programs. The curriculum now recommends that estimation be specifically taught within school mathematics programs, and this has influenced the attention given to estimation in current elementary school mathematics programs.

Yoshikawa notes that Japanese students are highly skilled in producing exact answers and reports that this ability may inhibit some students from estimating. Yoshikawa's paper highlights some of the difficulties Japanese students have in understanding estimation and in producing estimates, many of which are echoed in Trafton's characterization of American students. Efforts have been made to assess the estimation skills of Japanese students, and these results suggest that performance is quite low. Some recent data from several different assessments are reported in the papers by Yoshikawa and by Itoh and Mizusawa.

Two chapters (Ishida and Shimizu; Nohda and Yabe) report on two Japanese research studies focusing on estimation. Each study provides interesting and valuable insight into how children as well as adults learn and do estimation.

Cognition

Cognition and metacognitive processes relate to both mental computation and computational estimation. Sowder and Silver each provide a synthesis of the current state of development and some needed direction for change. Whereas Silver focuses on situated problems as a context for developing and studying mental computation and estimation, Sowder provides a valuable and useful overview of both cognition and metacognitive processes. She discusses some critical issues related to both thinking and learning and also presents some suggestions for future curriculum and instructional efforts. Sowder notes that a firm grasp of the computational process can help students become confident. This self-confidence allows them to believe that the processes of estimation and mental computation make sense and that the results are reasonable. This disposition toward sense-making is critical, she writes. The major difference between those who succeed in mathematics and those who don't is that good mathematics learners expect mathematics to make sense.

Calculators

The growing availability of hand calculators in the last quarter of the twentieth century has spawned another computational alternative to be accommodated in school mathematics. In less than twenty years, both countries have experienced an evolution from primitive, expensive electronic calculators to inexpensive hand calculators that are as powerful as a large computer a few years ago. There the similarity ends, since the support for calculator use in schools is different in the two countries.

In Japan, there has been little use of calculators in schools. The latest curriculum encourages calculator use in grades 5 and above but not in earlier grades. The current situation in Japan requires students to do computation without calculators in schools but assumes that they will make use of calculators outside of school. Senuma provides a current picture of calculator usage along with a rationale for the current situation.

In the United States there has been long and strong support for the use of technology in schools, and calculators in particular have been advocated by numerous reports from professional organizations. As early as 1975, the National Advisory Committee on Mathematical Education recommended that "beginning no later than the end of the eighth grade a calculator should be available for each mathematics student during each mathematics class. Each student should be permitted to use the calculator during all of his or her mathematical work, including tests" (Conference Board of the Mathematical Sciences 1975, p. 138). The National Council of Teachers of Mathematics addressed the importance of calculators in many of its publications and reports, including an official position statement that recommended the "integration of the calculator into the school mathematics program at all grade levels in classwork, homework, and evaluation" (NCTM 1986, p. 1). Coburn and Wheatley report that more students in the United States are using calculators in school than ever before; yet they also note that the extent of calculator use varies greatly within as well as across schools. Furthermore, calculators are often used in mundane ways instead of being used as a tool for promoting mathematical thinking and learning.

Instructional efforts related to the wise and effective use of calculators require that students recognize when to use a calculator and when other means of computation are more appropriate. Furthermore, the increased popularity of calculators creates additional pressures as well as incentives for mathematics programs to develop good mental computation and estimation skills, since these skills are essential in determining whether calculator results are unreasonable.

Instruction

Finally, another major issue centers on teaching. "How should instruction be organized to facilitate the learning of computation?" is a question critical to both countries. In both Japan and the United States there is a growing awareness of the need to furnish interesting and relevant contexts for doing computation. Consequently, the amount of attention devoted exclusively to the development of computational skills has decreased and is being replaced with more applications and problem solving, which furnish opportunities for computational alternatives. Silver's paper discusses this issue more fully as he develops the notion of "situated problems."

To what extent should mathematics be taught directly or result from students' constructing or inventing their own knowledge? This issue is more than a matter of semantics and is now being debated and discussed. To what extent should computational alternatives be taught directly to students as contrasted to providing situations for problems that allow students to "construct" their own strategies and algorithms? This important issue has far-reaching curricular and instructional ramifications. Care needs to be exercised to avoid creating an unnecessary, false dichotomy between directed instruction and constructivism. Reys and Barger, Yoshikawa, Trafton, Sowder, and Silver all provide further insight into ways of addressing issues of instruction.

A Focus on Research

In an effort to give a broad perspective for both researchers and practitioners, Richard Shumway was asked to identify and discuss critical research questions related to computational alternatives that need attention. His final chapter provides this discussion, and perhaps it will help stimulate a host of related research activities.

Part 1

Similarities and Differences: Mental Computation in Japan and the United States

Mental computation as a mathematical process can be viewed in various ways by educators. One view asserts that mental computation is a basic skill, perhaps serving as a prerequisite for paper-and-pencil computational skill or estimation, where proficiency is gained by direct teaching and practice. Another viewpoint proposes mental computation as a higher-order thinking process where the generation of a strategy is as important as the execution of the strategy. Japanese mathematics educators typically hold the former view of mental computation, whereas mathematics educators in the United States tend to have the latter viewpoint. The differing views in each country related to this topic result in differing instructional approaches and varied levels of emphasis on mental computation.

Viewed as a basic skill for computation or estimation, mental computation is seen as a set of procedures applied mentally. For example, computing the product of 40 and 60 mentally is a matter of multiplying 4 and 6 (a "basic fact"), then supplying a set of two zeros to reflect correct place value. Subscribers to this view recognize the importance of a carefully delineated and sequenced mental computation curriculum. Establishing what strategies (algorithms) will be taught and by what methods is of primary interest to them.

But when students generate their own computation strategies, mental computation can be seen as a higher-order thinking skill. In such situations, students are given meaningful problems that require computation. Rather than being taught standard algorithms (either paper-and-pencil or mental), students are encouraged to generate strategies for computing that are based on their intuitive understanding of the numbers and action needed. For example, to compute the cost of three erasers priced at 19 cents each, a second grader might think, "20 and 20 and 20 is 60, but that's 3 cents too much, so the cost is 57 cents." When used this way, mental computation is more than the mental application of an algorithm. The acts of generating a strategy and of applying it are crucial. This expanded view of mental computation is sometimes called "thinking strategies."

Viewing mental computation as a basic skill allows one to organize and sequence instruction so that it can be taught directly. However, viewing mental computation as a higher-order thinking skill requires major changes in curriculum organization, teaching philosophy, and instructional techniques. As a higher-order skill, instruction must be learner driven.

Historically, the Japanese view of mental computation is more closely associated with considering it a basic skill for computation itself. Although many Japanese mathematics educators, researchers, and cognitive psychologists support the alternative view of mental computation, they also think the changes suggested by this view of mental computation would be very difficult to implement in the national curriculum. As a result, the importance of the standardization of the curriculum takes a higher priority than an emphasis on student-generated algorithms.

In the United States, emphasis on mental computation as higher-order thinking is consistent with current reform movements, such as those suggested by the NCTM *Standards*. However, because the United States does not have a national curriculum, there are many variations in the role of mental computation in classrooms as well as in district curriculum guides. Although many American researchers and cognitive theorists are advocating the expanded view of mental computation, it is a relatively new way of thinking for many teachers and curriculum developers.

In summary, both Japanese and American educators recognize two ways of viewing mental computation. These options should not be considered mutually exclusive, since both have a place in mathematics curricula. Although Japanese and American educators see the potential of viewing mental computation as a higher-order thinking skill (where decision making is a key element), the widespread realization of such a perspective within classrooms in either country would require a major reorganization of the elementary school curriculum.

2

Computation in Japan from the Edo Era to Today: Historical Reflection on the Teaching of Mental Computation

Rokuji Shibata

THE Japanese terms associated with mental and paper-and-pencil computation (*anzan* and *hissan*) have a long history and are used differently than similar terms in the United States. Therefore, in this chapter, the meaning of *anzan*, which corresponds to mental computation, is clarified by tracing its evolution. Because anzan has always been related to hissan, which corresponds to paper-and-pencil computation, a brief survey of both terms will contribute to a better understanding of each country's perspective of mental computation.

EARLY COMPUTATION METHODS THROUGH THE EDO ERA

Arithmetic was imported to Japan from ancient China. For example, the famous Chinese arithmetic book entitled *Kyu sho Sanjutu*, which means "techniques of arithmetic," consisted of nine chapters and set the arithmetic curriculum through the Nara era (710–794). The text addressed a variety of topics and numbers, including fractions, square roots, and so on, and its instructional approach fostered computational facility through solving problems. The *soroban* (Japanese abacus) and *sangi* (Japanese counting blocks) were not yet being used very much. In summary, this classic text, which served as the mathematics curriculum of the day, was a collection of high-level mathematical techniques with an emphasis on applications.

In the beginning of the Edo era (1603–1867), the "shogun" (the chief samurai, who ruled Japan) closed trade and communication with other countries. During this period, Japanese arithmeticians had developed advanced calculation techniques, such as solving equations using sangi. However, in popular education schools (called *terakoya*, meaning "a private elementary school in the Edo period") private tutors taught arithmetic with the soroban. Furthermore, a famous Japanese textbook of the day, *Jinkou ki*, was devoted to teaching students computation using the soroban.

To use the soroban, it was necessary to remember the basic number facts of multiplication and division. Memorizing the addition and subtraction facts was not required, since these facts are easily obtained by moving *tama,* small pieces on a soroban, and people of the day were trained to use the soroban very quickly. Therefore at this time, mental computation was limited to recalling multiplication and division facts when using a soroban.

In summary, during the Edo era, computation meant using a soroban or sangi, and mental computation involving division and multiplication was necessary to execute this computation. There is no evidence to support the existence of what is currently called paper-and-pencil computation. Therefore, from a historical perspective, mental computation predates paper-and-pencil computation in Japanese society, and mental computation of this era was embedded in the use of the soroban.

MEIJI ERA: THE INTRODUCTION OF PAPER-AND-PENCIL COMPUTATION FROM WESTERN SOCIETY

Until about 1850 in Japanese educational history, paper-and-pencil computation as we know it today did not exist. In the beginning of the Meiji era (1867–1927), hissan (paper-and-pencil computation) was introduced from Western society. The term itself first appeared in *Seisan Sokuti* ("Quick Understanding of Western Arithmetic"), written by Riken Fukuda in 1857. In this book, *hissan* was used to mean algorithms of paper-and-pencil computation. The author's main source was the Chinese edition of A. Wyle's *Introduction to Mathematics.* In the Chinese edition, *hissan* was referred to as "mathematics introduced by Western people." Over time, *hissan* came to mean only the algorithms of computation in Western mathematics.

The first national syllabus in Japanese history, *Shogaku Kyosoku,* shows the influence of Western education. For example, the school subject including arithmetic was called *youhou sanyou,* meaning "Western techniques of arithmetic." In this subject, students learned the arabic numerals, the numerical system, and paper-and-pencil computation algorithms. Mental computation with addition and subtraction was also included in the first semester of first grade in elementary school. The soroban, which the Japanese had been using in computation, faded in the mathematics curriculum during this time. Although there was a serious lack of teachers who were prepared to teach Western mathematics, it was given priority treatment by the Japanese government. Learning hissan was viewed as a necessary step in the development of Japan. At the same time, the government wanted to keep mental computation in the course of study. A new term, *anzan*, meaning "mental computation," was prepared and first used in *Shogaku Kyosoku*. Until this time, a special term for mental computation had not been necessary, since it had been implied in the use of the soroban.

This was the birth of mental computation in the Japanese mathematics curriculum. In *Shogaku Kyosoku*, mental computation was defined as follows: Anzan is computation in *mune* (chest) and results in an exact answer by devising it mentally without tools. Anzan included computing the exact answer mentally, computing by using a mental image of a soroban, and computing mentally by using arithmetic properties (e.g., $98 + 47 = (98 + 2) + (47 - 2) = 100 + 45 = 145$). Another popular word, *munazanyou*, also means computation without tools, but it implies the expectation and anticipation of such computation.

It is interesting to speculate on the reasons why the Ministry of Education adopted the new word *anzan* for mental computation when they had soroban, which already included it. In the learning of basic facts and simple computations, it was much more convenient to use the soroban. If the use of the soroban was allowed at this time, learning paper-and-pencil computation (hissan) would not have been a necessary task for the Japanese and the new hissan curriculum of the day might have been defeated by the soroban, which had a long tradition. But the Japanese government was afraid of colonialism in this century, so it promoted the importation of Western science, technology, and military power. In these fields, Western mathematics is used, so hissan was introduced in mathematics curriculum under the direction of the government. In the schools, pupils learned hissan and also learned the Western spirit of science through it. However, paper was expensive at this time, so teachers taught pupils to compute mentally when the numbers were simple. This was the place of anzan in the mathematics curriculum during this time. Later, the soroban was also permitted for the same reason. By introducing hissan and anzan, teachers could incorporate the new Western mathematics and still retain the traditional Japanese soroban techniques, thereby establishing two methods of computation. Therefore, they created the term *hissan* for formal algorithmic computation and the term *anzan* for mental computation. Unfortunately, because of the new term, a false impression was given to the public that mental computation was different from the mental computation used with the soroban.

SHOWA ERA: STANDARDIZING THE CURRICULUM OF COMPUTATION

In the beginning of the Showa era (1927–1989), the computation curriculum was drastically modified by Naomichi Shiono, who was the textbook inspector of the elementary mathematics curriculum at the Ministry of Education. In the "green cover" textbook (1935), emphasis on anzan increased. The rationale for the higher position given to mental computation was that people usually compute mentally, and only when this is impossible do they compute with paper-and-pencil techniques.

In this new curriculum, mental computation was taught before paper-and-pencil computation algorithms. For example, the formal algorithm for adding three-digit numbers was taught in third grade, but mental computation for adding simple three-digit numbers was taught in second grade. The following was characteristic of the new curriculum:

1. Mental computation was given a prominent position in the mathematics curriculum.
2. A scope and sequence for mental and paper-and-pencil computation was developed, with mental computation preceding paper-and-pencil techniques.
3. Habituation (automatic recall and response) of mental computation procedures was recommended.

There were two principles of teaching mental computation: (1) Compute from "top to end" digit and (2) Say what you do in your mind. For example, when computing $28 + 34$, think: $28 + 34 \rightarrow 28 + 30 = 58 \rightarrow 58 + 4 = 62$, and say in your mind, "Twenty-eight and thirty make fifty-eight. Fifty-eight and four make sixty-two, so the answer is sixty-two."

The importance of these principles cannot be overstated. The first principle helps students grasp the general size (estimate) of the answer. It also gives them an opportunity to decompose numbers. The second principle helps students control their thinking on computational processes. Because of the clearly laid out scope and sequence and a prescribed instructional strategy, the program was widely adopted in classrooms. Although Shiono's direct influence ended after World War II, many of his ideas continue to influence mental computation today.

SOME THOUGHTS ON MENTAL COMPUTATION IN THE CURRENT *CURRICULUM OF MATHEMATICS FOR THE ELEMENTARY SCHOOL*

In spite of the spirit of the current course of study and the increased emphasis on estimation, the treatment of mental computation has been de-emphasized. Further, the attention to mental computation seems to be more rigid and strictly formalized than is necessary or helpful to students. With the introduction of the "thought experiment" in the new course of study, mental computation may play a new role. Thought experiments emphasize informal thinking in problem solving. Mental computation, if taught before formal paper-and-pencil algorithms, might promote insight into viewing mental computation as a framework for these formal procedures. The idea of viewing mental computation as informal thinking where students create procedures is new and will need more exploration in Japan.

3

Mental Computation: Evaluation, Curriculum, and Instructional Issues from the Japanese Perspective

Keiichi Shigematsu
Hideki Iwasaki
Masataka Koyama

Today is sometimes called the age of paper-and-pencil computation, since so much emphasis is placed on it in elementary school mathematics in Japan. In this age, mental computation is regarded as a basic computational skill for paper-and-pencil computation. Outside of school, however, many students and adults depend on a calculator for computation. Sometimes students even use calculators for finishing homework problems intended for paper-and-pencil computation.

Reys and Barger (chapter 4) suggest that mental computation is "a vehicle for promoting thinking, conjecturing, and generalizing." In this view, a major emphasis of instruction is on allowing students to devise their own computational methods. In Japan, however, it would be difficult to include this thought-building aspect explicitly in mental computation instruction because the Japanese curriculum expects all the teaching points to be attained to the same extent by all the students in the same period of learning time, leaving very little room for individual variation. Therefore, although we recognize the importance of developing students' higher-order thinking, we do not believe that mental computation is the appropriate vehicle.

Mental computation in the narrow sense refers to the computation process alone, which in turn involves two different subprocesses. One is needed when students carry out the automatic computation in the brain without the aid of any external devices. The other is needed when the student carries out paper-and-pencil or abacus computation in the brain without the aid of external devices. For example, when students, trying to compute 57 + 46, find it difficult to compute it automatically in the brain without the aid of any external devices, they usually visualize the process of its paper-and-pencil computation in the brain.

Mental computation in the broad sense, however, includes not only the computation process but also the higher-order thinking and decision making that lead to the selection of the computational process. In the near future calculators will replace paper-and-pencil computation algorithms as the main tool for computation. Mental computation in the narrow sense will be extremely important not only for checking the operations and results of calculators but also for monitoring, evaluating, and controlling the whole process of computation. The value of mental computation will thus increase in the age of the calculator.

In this paper, we trace the history of mental computation in the Japanese school mathematics curricula, examine instructional techniques of mental computation, assess students' cognitive performance on mental computation, and present a rationale for including mental computation in future Japanese mathematics curricula.

WHAT HAS HAPPENED?

A brief evolution of computation in Japan was provided by Shibata in chapter 2 in this volume. Although school mathematics in Japan is still dominated by written computation, the age of the calculator is dawning. The calculator is already the main computational means in Japanese daily life today, although not in schools. Nevertheless, the advent and increasingly widespread availability of hand calculators has presented a new and important perspective for computation not only in "street mathematics" but also in school mathematics.

Ironically, the widespread use of the hand calculator may increase the importance of mental computation. Because calculators show nothing about the process of computation, users must estimate the result to know whether the output is correct. This kind of estimation is often accomplished by rounding numbers, computing the rounded numbers mentally, then comparing the result to the calculator's output.

Such reflection is indispensable when using a calculator, and mental computation plays an essential part in the reflection process. This reflective use of mental computation is what we refer to as "metacomputation," or the conscious act of judging the result of computation.

Paper-and-pencil computation is based on the positional notation of numbers, whereas mental computation is based on the numeration system. In this way, mental computation is more often done in an oral mode but can be easily developed into a visual mode. Phonetic symbols for mental computation appear and soon disappear. Common mathematical symbols are not seen in oral mode. Therefore, we must teach students to respond quickly and exactly as follows:

1. Break numbers into small parts.
2. Compute from larger value (e.g., hundreds) to smaller value (e.g., units).

For example, the following computations would be performed mentally as shown:

1. $56 + 38 = (56 + 30) + 8 = 86 + 8 = 94$

2. $142 - 97 = (142 - 90) - 7 = 52 - 7 = 45$
3. $37 \times 4 = (30 \times 4) + (7 \times 4) = 120 + 28 = 148$

The Japanese numeration system is suitable for these above methods because of a base of ten that is obvious by its word names. Therefore, Japanese mental computation is aided by our own numeration system. The European numeration system, however, has several ways of presenting its numbers orally. For example, consider the expression 93 ± 18:

in Japan:	ku-jyu san ± jyu hachi $(90 + 3) \pm (10 + 8)$
in America:	ninety-three ± eighteen $(90 + 3) \pm (8 + 10)$
in Germany:	drei-und-neunzig ± achtzehn $(3 + 90) \pm (8 + 10)$
in France:	quatre-vingt-treize ± dix-huit $(4 \times 20 + 13) \pm (10 + 8)$

The methods of mental computation are closely associated with the unique oral language in each country. Our research and classroom experience suggests that the language used with the numeration system has much effect on the high level of achievement in mental computation by Japanese students.

There might be a common standard of mental computation in the visual mode because of the Hindu-Arabic notational system. In practice some Japanese children answered that they could compute in their heads with the image of the paper-and-pencil computation when we interviewed them in our research. Each of them rounded numbers adequately and devised an easy way to do so depending on their own capacity of computational image. A cross-cultural study of the use of this technique would be of interest.

The prevalence of the calculator causes our daily calculations to surpass the paper-and-pencil computation of school mathematics. Therefore, mental computation is freed from being a basic skill for paper-and-pencil computation. It is connected not only to checking the answer in street mathematics but also to developing in school mathematics the higher-order thinking skill and self-generated thinking strategies described by Reys and Barger. Mental computation is no longer a skill that is subconsciously developed but a metacomputation worth being consciously learned. It becomes a key ingredient for estimation.

WHAT IS HAPPENING?

In this section we shall describe the present situation with regard to mental computation in Japan and provide a base from which we can get a clear perspective of future directions. We also propose an outline of the future curriculum of estimation and mental computation. To do this, we first delineate and evaluate the present curriculum of mental computation and then analyze the instruction of it within textbooks. Finally, we summarize Japanese students' performance on mental computation and the metacognitive beliefs held by them and their teachers.

Curriculum and Instruction

According to the *Curriculum of Mathematics for Elementary Schools* used during the 1980s, it was necessary for children to practice mental computation to the following levels in the third grade:

1. Addition of 2-digit numbers, and subtraction as the inverse operation.
2. Multiplication of 2-digit number and 1-digit number, and division as the inverse operation. (Japanese Ministry of Education 1977)

However, in the current *Curriculum of Mathematics for Elementary Schools*, the level of mental computation in the third grade is not prescribed in detail. In fact, it states, "It is necessary to help children carry out simple computations mentally" (Nagasaki, Sawada, and Senuma 1990). The notion of "simple computations" is less clear in the current course of study. Therefore, it is an important educational issue in Japan to clarify the degree of simple computations to be carried out mentally. Once clarified, mental computation should be developed as an important alternative to paper-and-pencil computation in a highly technological society that values estimation.

In order to document the current emphasis on mental computation in Japanese textbooks, the second- and third-grade student editions of the six textbook series currently used in Japan were studied. All were written using the guidelines indicated by the current *Curriculum of Mathematics for Elementary Schools*. Although all textbooks include mental computation in four operations (i.e., addition, subtraction, multiplication, and division) with whole numbers, we identified a variation of emphasis on mental computation. In fact, only three textbooks included individual teaching units for mental computation with the four operations. Several examples of mental computation exercises with four operations cited from three textbooks are shown in table 3.1.

Table 3.1
Examples of Mental Computation Exercises in the Three Third-Grade Japanese Textbooks

Textbook	+	−	×	÷
Elementary School Arithmetic (1991)	35 + 26 39 + 28	53 − 27 158 − 95	25 × 3 32 × 4	84 ÷ 2 72 ÷ 3
New Arithmetic (1991)	54 + 38 540 + 380	54 − 38 540 − 380	25 × 3 250 × 3	74 ÷ 2 740 ÷ 2
Arithmetic (1991)	56 + 42 38 + 21	56 − 42 38 − 21	24 × 3 25 × 3	75 ÷ 3 52 ÷ 2

The teaching methods of mental computation in the three textbooks are a little different. In *Elementary School Arithmetic* (Hirabayashi 1991), the strategies of mental computation are only suggested by using the utterances of some students, as shown in figure 3.1.

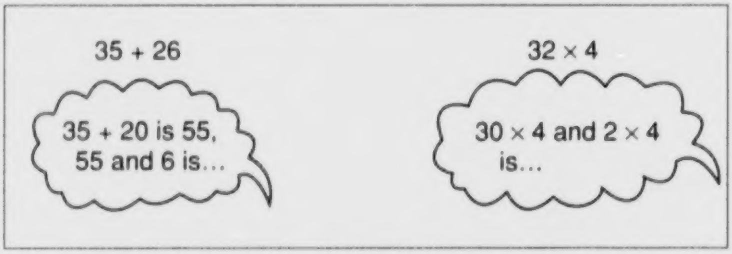

Fig. 3.1. Illustration of strategies adopted in the first Japanese textbook to develop mental computation

In the textbook *New Arithmetic* (Maehara 1991), the strategies of mental computation are explained using both the decomposition of the number into tens and units and the procedures of computation (see fig. 3.2).

In the textbook *Arithmetic* (Hosokawa and Sugioka), both the utterances and the decomposition are presented (fig. 3.3).

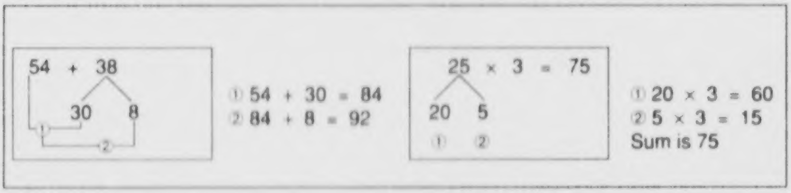

Fig. 3.2. Illustration of strategies adopted in the second textbook to develop mental computation

To summarize, except for two textbooks in which mental computation of addition and subtraction is taught in the second grade, mental computation with the four operations is taught in the third grade. The quality and quantity of instructional activities for mental computation included in the teaching units or exercises are similar. However, the teaching methods for mental computation are a little different among these textbook series.

In the previous course of study estimation was not highly emphasized. Therefore, the relationship between mental computation and estimation was not explicitly stated. However, mental computation does assist written computation. In the current course of study, estimation is emphasized to a greater degree, whereas the emphasis on mental computation is

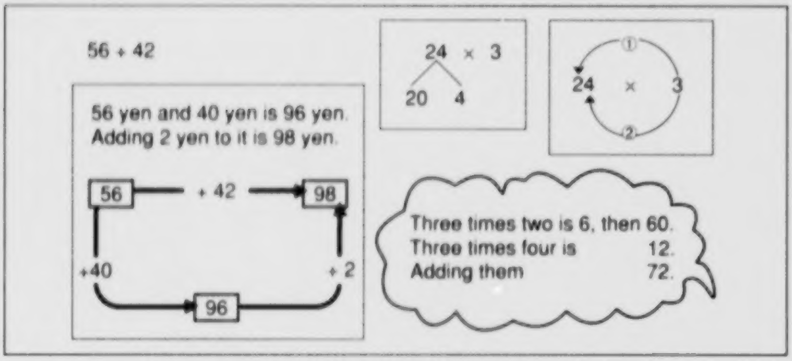
56 + 42
56 yen and 40 yen is 96 yen.
Adding 2 yen to it is 98 yen.
56
+ 42
98
+40
+ 2
96
24 × 3
20 4
①
24 × 3
②
Three times two is 6, then 60.
Three times four is 12.
Adding them 72.

Fig. 3.3. Illustration of model used by the third textbook to develop mental computation

diminished. In this ironic situation, we should evaluate mental computation as "metacomputation" for the calculator and as a basic skill for estimation.

Students' Performance on Mental Computation

Data on students' performance on mental computation in Japan is limited. In an attempt to gather performance data in this area, a study was conducted in June of 1990. The study focused on assessing the cognitive performance and the metacognitive beliefs on mental computation. Students in grades 3, 4, 5, and 6 (a total of 592 students) were selected from four different elementary schools located in Nara, Hyogo, and Hiroshima prefectures. Students were placed into one of two groups—one group was administered a written mental computation test, the other an oral mental computation test. In addition, the results of a questionnaire designed to document metacognitive beliefs was administered to the students and thirty-six elementary teachers.

Figures 3.4 and 3.5 present the mental computation tests administered to students in the third grade (MC-1) and to students in the fourth, fifth, and sixth grades (MC-2). Each test consisted of twelve items involving the four operations with whole numbers. At each grade, these test items were presented either in a written test or in an oral test.

In the written test, all items were included on a single sheet of paper distributed to students with the instruction that they mentally compute exact answers for all problems. They were given ten minutes to complete the written test. For the oral test, all items were prerecorded on a recorder and presented to students at a specified period of time (five to ten seconds an item) with the instruction that they compute mentally

Compute the following exactly without using paper and pencil (i.e., mentally). Write an exact answer in the frame ☐. If you cannot compute mentally, make an X in the frame. [Frames are omitted.]

1. 78 + 21
2. 43 + 58
3. 357 + 32
4. 209 + 542
5. 16 − 3
6. 53 − 8
7. 42 − 13
8. 145 − 46
9. 345 − 197
10. 7 × 8
11. 15 × 4
12. 324 × 2

Fig. 3.4. MC-1 test administered to third-grade students

Compute the following exactly without using paper and pencil (i.e., mentally). Write an exact answer in the frame ☐. If you cannot compute mentally, make an X in the frame. [Frames are omitted.]

1. 43 + 58
2. 357 + 32
3. 209 + 542
4. 53 − 8
5. 42 − 13
6. 145 − 46
7. 15 × 4
8. 324 × 2
9. 54 ÷ 9
10. 96 ÷ 8
11. 603 ÷ 3
12. 284 ÷ 4

Fig. 3.5. MC-2 test administered to fourth-, fifth-, and sixth-grade students

exact answers (without copying the problem) and write their answer on a sheet of paper provided. The total test time for the oral test was therefore much shorter than for the written test.

Table 3.2 includes items from each test with accompanying performance levels in each grade. The number of students providing these data is also reported in table 3.2. Figures 3.6 and 3.7 display the same data in a bar graph.

From these results, we point out the following findings:

- *Performance on both the written and the oral tests generally improved with grade.* For example, although there was little difference in performance on items such as 43 + 58 or 53 − 8, in the cases of 42 − 13, 145 − 46, 15 × 4, and the division items, the percent correct increases gradually with grade level.
- In grades 4 and 5, *performance on the written test varied by operation,* with addition being the easiest and division the most difficult. This fact reflects both the emphasis on various operations in each grade as well as the emphasis given to mental computation in grade 3.
- There was a significant difference between the performance on the written and oral test. This difference likely results from a variety of factors including the format of presentation and timing of each test.

Table 3.2
Percent Correct on Mental Computation Problems

	Written Test				*Oral Test*		
Problem	3rd (*n* = 69)	4th (*n* = 92)	5th (*n* = 68)	6th (*n* = 83)	3rd (*n* = 71)	4th (*n* = 70)	5th (*n* = 75)
78 + 21	94				72		
43 + 58	87	96	96	98	48	60	69
357 + 32	84	92	99	96	34	59	65
209 + 542	83	90	99	95	18	30	32
16 – 3	93				93		
53 – 8	91	95	97	94	68	83	88
42 – 13	70	83	94	94	35	46	69
145 – 46	70	82	93	88	46	61	64
345 – 197	48				6		
7 × 8	96				96		
15 × 4	55	79	91	90	44	61	87
324 × 2	67	90	94	94	45	56	79
54 ÷ 9		84	91	99		73	87
96 ÷ 8		62	82	93		41	76
603 ÷ 3		60	78	94		56	83
284 ÷ 4		65	74	92		37	71

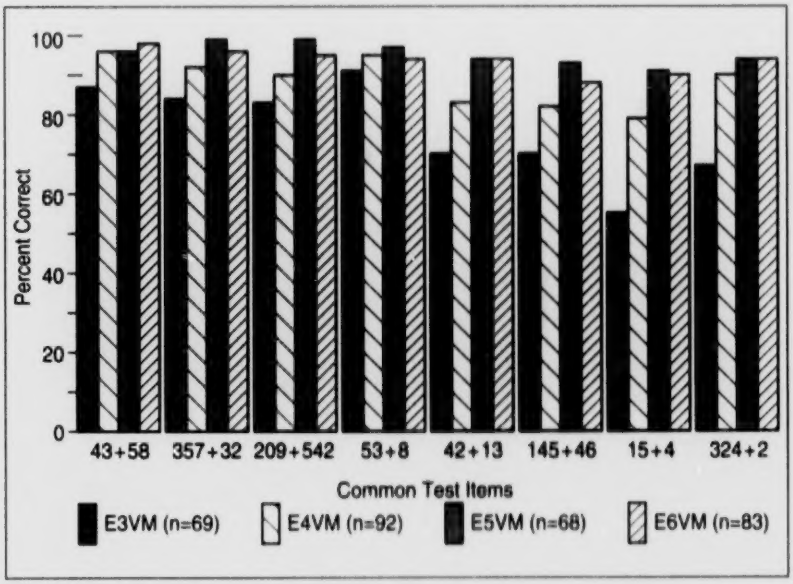

Fig. 3.6. Comparison of performances on mental computation in visual mode

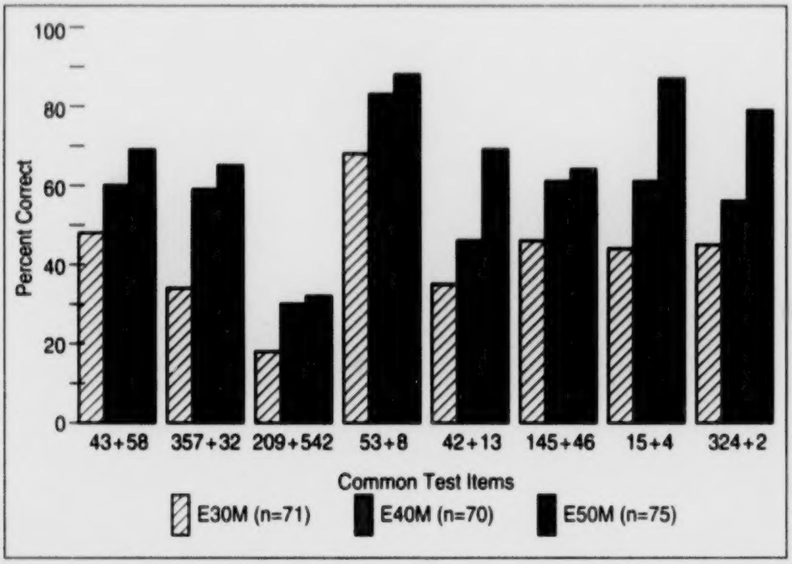

Fig. 3.7. Comparison of performances on mental computation in oral mode

> Other factors that may have contributed to better performance on the written test include a decreased need to "memorize" numbers. This conjecture is supported by the performance on the item 209 + 542. If students had used a mental version of the paper-and-pencil algorithm, it would be difficult to memorize the numbers and carry out this strategy in only five to ten seconds. However, if they had computed the problem mentally by using the *tohka-ho* strategy (e.g., 209 + 500 + 42 = 709 + 42 = 751), the difference in performance might not have been so great.

In summary, performance on mental computation with four operations and whole numbers improves with age. Many students, however, seem to employ a mental version of the paper-and-pencil algorithm instead of making use of the advantages of the Japanese numeration system.

Metacognitive Beliefs of Teachers and Students

The questionnaire in table 3.3 focused on the students' and teachers' attitudes and beliefs about mental computation in a school of Nara prefecture. Table 3.4 presents a summary of the results. From these data, we can point out the following findings:

- Teachers and students think that elementary school mathematics is an exact and strict subject and therefore doubt the value of estimation.
- Some teachers acknowledge that mental computation is an important procedure for estimation but do not emphasize mental computation. They attribute this to the fact that they have no explicit syllabus for teaching mental computation.
- Third- and fourth-grade students tend to be more positive about the use of mental computation than fifth- and sixth-grade students. This may be because the level of computation at the lower grades is rather simple.

Table 3.3
Questionnaire on Students' and Teachers' Metacognitive Beliefs

Q1: It is important for students to compute exactly in elementary school mathematics.
Q2: Mental computation is important for estimation.
Q3: It is appropriate for students to be able to use mental computation for 2-digit numbers ± 1-digit numbers.
Q4: It is appropriate for students to be able to use mental computation for 2-digit numbers × / ÷ 1-digit numbers.
Q5: I will teach mental computation more for students to be able to estimate well.
Q6: I have a good performance of mental computation.
Q7: I always think that I must give an exact answer in elementary school mathematics.

Table 3.4
Results of Questionnaire

		Percent of Each Response			
Number of Subjects		Strongly Disagree	Slightly Disagree	Slightly Agree	Strongly Agree
Q1	Teachers (*n* = 12)	0	0	42	58
Q2	Teachers (*n* = 11)	0	9	36	55
Q3	Teachers (*n* = 12)	8	25	50	17
Q4	Teachers (*n* = 12)	0	25	58	17
Q5	Teachers (*n* = 12)	0	33	50	17
Q6	Teachers (*n* = 11)	27	27	45	0
	3rd Graders (*n* = 56)	13	21	41	25
	4th Graders (*n* = 64)	11	25	34	30
	5th Graders (*n* = 75)	24	32	29	15
	6th Graders (*n* = 86)	23	35	33	9
Q7	Teachers (*n* = 11)	0	18	9	73
	3rd Graders (*n* = 56)	9	11	16	64
	4th Graders (*n* = 64)	6	6	22	66
	5th Graders (*n* = 75)	5	15	21	59
	6th Graders (*n* = 86)	5	15	28	52

FUTURE DIRECTIONS

The current national curriculum for elementary school gives estimation greater attention. We believe this is a step toward moving to the age of the calculator.

In the near future we believe the calculator will become the main computation tool, replacing the current use of paper-and-pencil computation. Therefore, mental computation will become more than merely a basic skill for paper-and-pencil computation but rather a metacomputation for computation by the calculator. In this light, we predict that students will depend more than now on calculators in school as well as at home, even when they solve a simple arithmetic problem, and that as a result they will come to depend less on tedious paper-and-pencil computation.

Since the column form of paper-and-pencil computation provides a visual image for mental computation, paper-and-pencil computation will become a basic skill for mental computation rather than the other way around.

In the transitional stage between paper-and-pencil computation and calculators, there will likely be confusion in curriculum, instruction, and assessment. Japanese elementary school teachers traditionally consider paper-and-pencil computation with the four basic operations and whole numbers the main objective of mathematics instruction. Moreover, it has been required in Japan for children to be able to compute exactly and automatically with paper and pencil as if they were calculators. However, in the future we don't think children will be required to become calculators. Instead, children must learn both how to use calculators and how to estimate answers as they solve problems requiring computation.

In table 3.5, we present a rough outline of the curriculum for paper-and-pencil and mental computation.

Table 3.5
Curriculum of Mental and Paper-and-Pencil Calculations

Grade	Mental				Paper-and-Pencil			
	+	−	×	÷	+	−	×	÷
1	facts	facts						
2			facts		III + II	III − II		
3				facts			II × I	II ÷ I

Note: I = 1-digit numbers; II = 2-digit numbers; III = 3-digit numbers

In general, we envision four stages in the development of computation in the elementary school curriculum. In the first stage, children learn basic facts for addition, subtraction, multiplication, and division. In the second stage, they learn paper-and-pencil computation to the following level: addition and subtraction with two- and three-digit numbers and multiplication and division with one- and two-digit numbers. In the third stage, they mentally carry out paper-and-pencil algorithms (this is because not all students can do mental computation automatically or by using *tohka-ho*; therefore, they can perform mental computation using the paper-and-pencil algorithms they have already learned). In the fourth stage, they apply mental computation to estimation. With this progression, they will be able to use the calculator efficiently and with confidence because they have tools to reflect on the calculated output.

Regarding methods of evaluating children's performance on mental computation, we are not satisfied with current assessment methods because mental computation is a mental process that we cannot see directly. Therefore, we must develop procedures that discourage students from doing paper-and-pencil computation and combine some methods such as using questionnaires, microcomputer assessment, and interviews of individual students. Cross-cultural research studies in this area are greatly needed (Reys, Reys, Nohda, and Emori forthcoming).

4

Mental Computation: Issues from the United States Perspective

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Mental computation, the process of carrying out arithmetic calculations without the aid of external devices, has assumed increasing importance in professional dialogues on school mathematics. This recent surge of interest in establishing mental computation in the mathematics curriculum is not a new phenomenon. In fact, emphasis on mental computation has a long and sporadic history within the school mathematics curriculum in the United States. The novel facet of this revival is the interest in using mental computation as a *vehicle* for promoting thinking, conjecturing, and generalizing based on conceptual understanding rather than as a set of skills that serve as an *end* of instruction. The contemporary view of mental computation, in sharp contrast to current methods of developing paper-and-pencil computation procedures, calls for mental computation (or thinking) strategies to be developed from conceptual understanding rather than from memorized rules. In this view, mental computation is characterized by a *manipulation of quantities*, whereas conventional paper-and-pencil algorithms encourage the *manipulation of symbols* (Reed and Lave 1981).

Plunkett (1979) contrasts written and mental processes. He characterizes written algorithms as *standardized*, *contracted* (summarizing several steps involving distributivity and associativity), *efficient*, *automatic*, *symbolic*, *general*, and *analytic* (requiring numbers to be broken up and digits dealt with separately). Written algorithms encourage cognitive passivity and are difficult to internalize because they do not correspond to ways people naturally think about numbers. In contrast, Plunkett characterizes mental algorithms as *fleeting*, *variable*, *flexible*, *holistic*, *constructive*, and *requiring understanding*. Further, he describes mental algorithms as *limited* (because they cannot be applied to all problems) and *active* (because users choose a method and control calculations).

Although dialogue has continued over the past century concerning the place and importance of mental computation in the classroom, surprisingly little research is available to guide either curriculum or instruction. Evaluation techniques used to document student performance on mental computation have been limited to timed tests administered to whole-class groups

This paper will briefly trace the history of mental computation in the school mathematics curriculum, examine instruction and assessment techniques as well as past and present student performance on mental computation, and explore the rationale for including mental computation in future mathematics curricula in the United States.

WHAT HAS HAPPENED?

Curriculum and Instruction

Throughout the history of American education, interest in mental computation has been a recurring theme. According to Smith (1913), around the middle of the nineteenth century Pestalozzi in Europe and Colburn in the United States began a revolt against the slowness with which students carried out written arithmetic. This revolt brought about an emphasis on "oral" arithmetic (problems that were orally stated and mentally figured) in American schools. Both student and teacher evaluations were based, in part, on students' proficiencies in mental arithmetic.

At that time, the rationale for including mental arithmetic in the curriculum centered on its use in improving the mind through mental discipline. The mind was thought of as a muscle, which needed exercise to become strong, and mental activity, especially the activity required in mental arithmetic, was viewed as particularly valuable for this purpose. Mental arithmetic often involved intense oral drill on complex computation problems involving several steps (Buchanan 1978). The following examples of "mental" problems are taken from a textbook of that period for upper elementary school students, *Higher Arithmetic*, by Walsh (1896), and help illustrate the extreme to which skill in mental arithmetic was expected and practiced. Similar problems were included in every section of the book as a regular mental drill feature.

> Bought 2 chairs at $1.25, 1 wash-tub for $1.50, 1 table for $3.00, and 5 doz. glasses at 48 cents a dozen. Gave a ten-dollar bill in payment. How much change did I receive?
>
> What profit is made on an article bought for $175, less 12%, and sold for $200?
>
> Adam lost $22. He then found 1/4 as much as he had remaining, and then he had 1/3 as much as he had at first. How much had he at first?

By the early 1900s, the emphasis on mental computation was influenced by a counterreaction in the philosophy of education. At this time, the theory of mental discipline fell into such disrepute that it was difficult to maintain a place in the curriculum for any form of mental activity, including mental arithmetic.

Until this time, few educators had envisioned a balanced curriculum, which would place appropriate emphasis on both mental and written computation. One far-sighted educator, however, proposed exactly that, saying,

> It is altogether probable that many simple calculations or analyses can be done "silently" from the beginning; that others require visual demonstration, but once mastered can thereafter be done without visual aids; that still others will always be performed, partially at least, with some written work. It is purely a matter for concrete judgment in each special case, but the existing practice scarcely recognized this truth. The result is that many problems are arbitrarily done in one way, and it is too frequently the uneconomical and inefficient way that is used. (Suzzallo 1911, p. 78)

Similarly, Thorndike (1922) argued that the two types of computation should supplement each other rather than compete:

> Devotion to orallness or mentalness as such is simply fanatical.... The common practice of either having no use made of pencil and paper or having all computations and even much verbal analysis written out elaborately for examination is unfavorable for learning... There is nothing intellectually criminal about using a pencil as well as inner thought; on the other hand there is no magical value in writing out for the teacher's inspection figures that the pupil does not need in order to attack, preserve, verify or correct his result. (Pp. 263–64)

The admonitions of these educators were generally unheeded. The early 1900s saw little or no emphasis on mental computation and returned to giving primary emphasis to written algorithms.

Between 1930 and 1960, the concern again emerged that pupils were becoming too dependent on paper-and-pencil algorithms and that more emphasis on mental arithmetic was needed. The increased emphasis on mental computation coincided with an attempt to generally improve instruction in mathematics. It was during this time that William Brownell pioneered efforts to emphasize meaning in the learning and teaching of mathematics. According to Brownell (1935), meaningful learning helps "make arithmetic less a challenge to the pupil's memory and more a challenge to his intelligence" (p. 31). In 1959, Max Beberman wrote in his introduction to *Teaching the Third R*:

> Mental arithmetic is one of the best ways of helping children become independent of techniques which are usually learned by strict memorization.... Moreover, mental arithmetic encourages children to discover computational shortcuts, and thus to gain deeper insight into the number system. Mental arithmetic also makes possible the learning of mathematical concepts at a very early age, since the child is not burdened by the intricacies of written symbolism. (P. 4)

The revival of mental computation peaked again just prior to the "new math" era of the 1960s. However, with the advent of the "new math" curriculum, the issue of paper-and-pencil versus mental computation took a backseat to learning mathematical structure. This de-emphasis of mental computation was ironic, since most proponents of mental computation today cite better understanding and use of structural relationships with numbers as a benefit of mental computation (R. Reys 1984). In the 1960s there was also a shift in the philosophic orientation of school mathematics away from drill and practice to discovery learning and independent inquiry, which ironically were not seen as compatible with any serious attempt to develop mental arithmetic as a usable skill (Buchanan 1978).

Research

In 1961, Klaas Kramer began collecting data on the effects of implementing a systematic approach to teaching mental computation strategies. After developing a series of lessons focusing on such strategies, he carried out a thirty-week comparative study with suburban fifth graders. The experimental group received instruction in mental computation on Mondays and Wednesdays and practiced those tasks on Fridays. No change in mathematics instruction was made in the control group. The data demonstrated that pupils who had used the mental computation materials improved their accuracy and increased their speed in mental computation (without increasing the total amount of time spent on arithmetic). Moreover, no difference between the experimental and control groups was found in performance on a standardized paper-and-pencil arithmetic achievement test, even though time was taken from the arithmetic class period of the experimental pupils for special instruction in mental computation (Kramer 1965). Similar results have also been reported by Schall (1970).

More recently, Markovits and Sowder (1988) studied the effects of mental computation instruction over a three-month period by interviewing fourth- and sixth-grade students before and after the treatment. Their instructional philosophy treated mental computation as higher-order thinking, emphasizing number and operation concepts rather than showing rules for mentally computing various problem types.

The researchers found a significant decrease in the use of standard algorithmic methods and an accompanying increase in the use of nonstandard methods at both grade levels after the instructional sequence. Whereas before the instruction, students' methods for mentally computing were primarily paper-and-pencil algorithms performed mentally, these students made use of alternative "mental"strategies after the instruction. The researchers believe that changes in methods indicate an increase in students' number sense.

It is interesting to note that only two of the four teachers who participated in the study were judged by the researchers to have implemented the spirit of the treatment, teaching the lessons in a manner that encouraged the exploration and discussion of different strategies (the interview sample was confined to students in these teachers' classrooms). The research may indicate that instruction of this type requires teaching skills not necessarily associated with teachers who are successful in teaching paper-and-pencil algorithms.

Assessment

Despite its perceived importance, few efforts have been made to assess performance levels on mental computation exercises. The only large-scale assessment effort was undertaken by the National Assessment of Educational Progress (NAEP) in its third assessment of mathematics, conducted in 1982–83. A pool of eighteen whole-number exercises was generated and administered orally in order to closely monitor time per exercise. Depending on the problem, students were given from nine to thirteen seconds for each item. All exercises were read only once, and answers were recorded on a separate answer sheet. Because all the NAEP mental computation exercises are unreleased, the data presented in table 4.1 are parallel exercises, similar to the unreleased exercises.

Although the exercises were limited to whole numbers, the data provide useful baseline information. An examination of the performance results reveals the following (R. Reys 1985):

- Performance of 9-year-olds was consistently below 50 percent correct.
- Performance improved with age.
- Wide ranges of performance occurred at each age level.
- Performance varied greatly by operation, with addition being the easiest and division the most difficult.

These data suggest that, for 9-year-olds, the ability to compute mentally is only beginning to emerge. For example, only about half the 9-year-olds correctly computed 64 + 20 mentally, whereas on a similar exercise given with the aid of paper and pencil, almost 90 percent correctly computed the result (Carpenter et al. 1984). These students spend most of their school mathematics time learning paper-and-pencil algorithms, and this emphasis may interfere with their ability to deal more abstractly with numbers. When the opportunity to use the algorithms is eliminated by

Table 4.1
Percent Correct on Mental Computation Exercises from the Third National Assessment of Educational Progress

	Age (in years)		
Exercise	9	13	17
6 + 47	47		
64 + 20	52		
40 + 50		86	96
36 − 9	20		
49 − 16		77	86
73 − 23	29		
700 − 600		92	97
1,250 − 400		39	57
4 × 30		73	88
2 × 34	25		
90 × 3	23		
60 × 70		46	55
15 × 20		35	57
4 × 625		25	43
20 ÷ 5		88	94
600 ÷ 15		32	58
3,500 ÷ 35		39	63
70 + 40 + 70 + 50		50	47

Note: Time per exercise ranged from 9 to 13 seconds

the removal of paper and pencil, students are left with no learned strategy to apply.

For 13- and 17-year-olds, the ability to mentally compute addition and subtraction exercises involving multiples of 10 is very strong. However, performance drops considerably when the computation requires multiplication, division, numbers involving nonzero digits, and more intuitive mental algorithms for place-value decisions. For example, only 25 percent of 13-year-olds and 43 percent of 17-year-olds correctly computed 4 × 625 mentally. Likewise, only 39 percent of 13-year-olds and 63 percent of 17-year-olds correctly computed 3,500 ÷ 35 without paper and pencil. Most errors for this exercise in both age groups dealt with place value. As with younger students, the data indicate strong reliance on learned paper-and-pencil algorithms. As the complexity or number of digits increases, it appears that students' reluctance to give up standardized paper-and-pencil algorithms inhibits their ability to successfully compute mentally.

Several researchers have gone beyond performance data, such as those reported by NAEP, to focus on characterizing the processes used by both high- and middle-ability mental computers. Barbara Reys identified a group of seventh- and eighth-grade students in each category on the basis of their performance on a mental computation screening test. They were interviewed individually and asked to mentally derive answers to various computation tasks and then to explain the strategy they used. The high-ability mental computers made extensive use of number properties and equivalent number forms to translate exercises into ones that were easier to compute mentally. The specific techniques they used included number properties (commutativity, associativity, distributivity); knowledge of, and flexibility with, place value; and connections among operations. Middle-ability mental computers, however, tended to use only a mental version of traditional paper-and-pencil algorithms to do each computation. The interviews documented these students' overreliance on prescribed, algorithmic techniques. These subjects seemed unable, or at least unwilling, to formulate mental strategies on their own. They viewed mathematics as a set of learned rules to be followed rather than as a flexible subject whose tools allow for creativity (B. Reys 1985).

At about the same time, Hope carried out a similar investigation characterizing the strategies used by fifteen skilled and fifteen unskilled eleventh and twelfth graders to mentally compute multiplication exercises. Introspective reports were used to document four methods of solution and twelve calculative strategies. As in the former study, Hope found that the unskilled students used strategies more suited to written than to mental computation and that the skilled students used strategies based on the number properties suggested by the factors in the exercise (Hope and Sherrill 1987).

WHAT IS HAPPENING?

Curriculum and Instruction

Recent years have seen a revival of interest in mental computation. This interest is more broadly based in the mathematics education community than it has ever been in the past. The reasons for renewed interest in mental computation are varied.

First, it coincides with calls for decreased attention to paper-and-pencil computation. Although most educators admit that the role and extent of paper-and-pencil facility has changed dramatically, that does not mean that the need for multiple methods of computation is diminishing. Instead, alternatives, including mental computation, calculators, and estimation, should also be employed when appropriate. Students need experience with each of these methods of calculation. NCTM's *Agenda for Action* (1980) was an early document to call for this balance of computational procedures. Reminiscent of Suzzallo's writing in 1911, it said, "Common sense should dictate a reasonable balance among mental facility with simple basic computations, paper-and-pencil algorithms for simple problems done easily and rapidly, and the use of a calculator for more complex problems or those where problem analysis is the goal and cumbersome calculating is a limiting distraction" (p. 6).

Second, the acceptance of the constructivist approach to learning has caused some mathematics educators to look at that theory's implications for the teaching of mathematics. Constructivists view the learner as the instigator and primary director of her or his own learning through the assimilation and accommodation of new and prior knowledge and experiences. Because self-generated strategies for arithmetic computation are often mental strategies, these educators believe that mental computation has a natural place in the classroom (Cobb and Merkel 1989).

Finally, mathematics educators agree that a chief benefit of mental computation, in addition to its practical applications, is its ability to help children learn to think. Rathmell and Trafton (1990) recognize the rich reasoning strategies encouraged by mental computation when they write,

> The varied and thoughtful ways children manipulate quantities when doing mental computation promote number sense as well as mathematical thinking. This kind of activity brings a dynamic quality to learning mathematics because children are actually *doing* mathematics rather than learning to repeat conventional procedures. (P. 157)

Lappan and Schram (1989) view mental computation as a vehicle for the exchange of ideas:

> Areas of mathematics that are now receiving increased attention… [such as] mental computation… have the additional advantage of being the source of particularly appropriate and natural experiences that are accessible to all children for developing communication and reasoning skills in the process of making sense of the mathematics. Asking children to describe what they think is going on in a mathematical situation… can turn a routine lesson into a "learning to think" lesson. (P. 29)

Cobb and Merkel (1989) believe the development of thinking strategies is evidence of increased conceptual understanding. They also cite important changes in beliefs about mathematics and further learning in students who have been encouraged to develop thinking strategies. "For them [project students], mathematics is about understanding, not just producing a page of correct answers" (p. 77).

Instructional Approaches

Although the benefits of a curriculum that encourages mental computation are generally agreed on, a preferred methodology of promoting mental computation is not evident. Currently, there are at least two ways of viewing the instructional treatment of mental computation in the elementary school curriculum, each supported by limited research. The first is to view mental computation as a "topic" to be delineated into identifiable strategies that are directly taught to students. This approach is similar to the traditional teaching of paper-and-pencil computation algorithms. Although the work of Kramer (discussed earlier) documents the short-term effects of this approach, most educators are skeptical about defining a new list of algorithms to be added to the present curriculum. They argue that mental computation will then lose many of the attributes outlined by Plunkett (fleeting, flexible, active, constructive) and might be more accurately described by his characterization of written computation.

The second approach is constructivist. Students are encouraged to generate thinking strategies based on their prior experience and knowledge. In this approach, students plan ways of solving problems as they arise in natural settings or as part of a planned problem-solving experience. Indeed, there is ample evidence that even young students, given an appropriate classroom environment, devise ingenious methods to solve arithmetic problems. Shuard (1987) relates a classroom incident involving six- and seven-year-olds who were part of an experimental classroom where emphasis was on understanding number rather than the ability to compute with numbers by predetermined methods. The class was asked to work out 28 + 29 in their heads and tell how they had done it. An adult in the classroom recorded the solution methods in symbols as the children described what they had done. Among the methods used and described were the following (p. 15):

1. Two children counted in ones, in different ways, using fingers.
2. 30 + 30 = 60; 30 − 29 = 1; 30 − 28 = 2; 1 + 2 = 3; 60 − 3 = 57.
3. 8 + 9 = 17; 20 + 20 = 40; 40 + 17 = 57.
4. 4 × 10 = 40; 40 + 9 = 49; 49 + 1 = 50; 50 + 7 = 57.
5. 25 + 25 = 50; 4 + 3 = 7; 50 + 7 = 57.
6. 20 + 20 = 40; 40 + 10 = 50; 8 − 1 = 7; 50 + 7 = 57.
7. 29 + 10 = 39; 39 + 10 = 49; 49 + 8 = 57.

Although it is clear that some students can formulate and use a variety of strategies, both elegant and not so elegant, the likelihood of their making use of and valuing such self-generated strategies seems to be closely tied to their notion of what school is about and in particular what mathematics is about. This is perhaps most evident in the work of Carraher, Carraher, and Schliemann (1987) who found that

> Situations that present quantities embedded in a meaningful transaction—such as calculating the amount of change after a purchase or the number of children in a school—seem to engage children in problem-solving procedures

> of the manipulation-of-quantities type.… In contrast, computations of the sort encountered in many classroom exercises seem to elicit written algorithmic solutions. This approach to problem solving leads children to focus on the written symbols, thereby losing track of both the meaning in the transactions they are quantifying and the meaning within the quantification system. This loss of meaning is likely to be the explanation for children's willingness to accept results such as a remainder larger than the minuend—a result that is promptly rejected if the child refers back to the meaning encoded by the particular computation. (P. 95)

The Teacher's Role

The teacher's role in developing mental computation varies according to the instructional approach employed. In one case, the teacher must organize and present a variety of strategies, then provide appropriate time for students to "learn" each strategy. In the other case, the teacher spends more time identifying appropriate problem contexts that will elicit different strategies. Clearly, the latter case requires teachers to be highly skilled in the art of questioning. Further, the teacher must have sufficient mathematical knowledge to examine student responses that, on the surface, may appear to be incorrect but may actually provide a springboard for a rich exploration into mathematical thinking.

Texts

While professional dialogue regarding mental computation increases, the classroom time devoted to it is typically sporadic and often misinterpreted. To document the current attention paid to mental computation in the mathematics curriculum, the second- and fifth-grade student editions of several popular textbook series (copyright dates ranging from 1987 to 1992) were studied. In each text, both the quantity and the quality of entries were examined to determine the number of times mental computation was specifically included in the text and the methods used in presenting it.

At the second-grade level, very little attention was given to mental computation. Only three to four activities a book were included, usually in the form of "challenge" or "enrichment" pages appearing at the end of chapters. Often they consisted of nothing more than instructions to "use mental math or paper and pencil" or "use any strategy you're comfortable with." In most instances, the exercises were designed in such a way that students could recognize a pattern and solve the problems without developing any understanding and without performing any real mental computation.

At the fifth-grade level, there were few complete lessons devoted to mental computation, though more attention was given to the topic than in the second-grade books. The lessons focused on topics such as multiplying by powers of 10 and multiples of powers of 10. Direct instruction —outlining a method for completing each problem—was virtually the only approach employed.

In general, the instructional activities related to mental computation varied greatly among textbook series, with most texts presenting the topic in a haphazard, unorganized fashion. The emphasis given to the topic within a single textbook series often varied from grade to grade with no indication of the rationale for this variation. Only one series attempted to integrate mental computation with other computation alternatives. Furthermore, there seemed to be no clear philosophy guiding the inclusion of the development of mental computation. Although most of the activities used direct instruction as the method of delivery, a few were open-ended and offered no prescribed method of solution. The placement of the activities further demonstrated that no philosophy guided its inclusion. Usually, mental computation activities were included after paper-and-pencil techniques had been introduced and developed, although in some instances they were mixed with exercises where students were expected to use paper-and-pencil procedures. Such evidence suggests that mental computation is often considered an appendage by textbook authors, receiving minimal treatment, when it should be viewed and treated as an integral part of mathematics learning.

Assessment

Assessment tools measuring student performance in mental computation have been limited to timed tests where individual items are presented for a specified period of time and students are asked not to copy the problem but to compute mentally. Some assessments, such as NAEP, have employed an oral presentation of the problems; others use either a visual mode (one item on each page or individual items projected on a screen) or a combination of both presentation modes (B. Reys 1985; Hope and Sherrill 1987; Reys and Reys 1993; Reys, Reys, and Hope 1993; Reys, Reys, Nohda, and Emori forthcoming). These studies have documented that student performance is often affected by the mode of presentation. Although there is no clear evidence that one mode of presentation is easier, there is an indication that for certain problem types students perform better when the format of the presentation is either oral or visual.

One study assessed the mental computation ability of students in grades 2 and 5 using both visual and oral presentation modes (Reys, Reys, and Hope 1993). At each level, a mental computation test consisting of both straight computation and applied computation items appropriate for the grade level was used. The tests included items presented orally followed by a second set administered visually using an overhead projector. Presentation followed a timed sequence of six to ten seconds for each item. The sample consisted of sixteen average-ability classes at both grade levels selected from two suburban school districts. The performance level for the second-grade sample ranged from 1 to 90 percent and for the fifth grade from 1 to 64 percent.

Table 4.2 includes several items from each test with accompanying performance levels. The data are consistent with previous NAEP data and generally show a pattern of increased difficulty as the numbers get larger or diverge from multiples of 10.

Table 4.2
Percent of Second and Fifth Graders Answering Correctly on Selected Mental Computation Items

	Grade	
Item	2 (*n* = 261)	5 (*n* = 250)
6 + 10	72	
20 + 40	43	
What is 50 more than 27?	2	
42 + 30	18	
80 − 30	32	
85 − 70	9	
42 − 29	1	
How much do 2 of these cost? (16-cent pencil pictured)	16	
How much are these nickels worth? (8 nickels pictured)	3	
300 × 40		30
8 × 99		26
50 × 22		10
25 × 28		6
3,500 ÷ 35		16

The researchers suggest that the low mental computation performance indicated a lack of attention paid to mental computation in the classroom. Few teachers of the students participating in this research reported spending much time developing mental computation, and most of them reported the lack of practical instructional ideas as a rationale for their limited or nonexistent efforts. Since none of the students had previously taken a mental computation test, the format of this timed test no doubt also contributed to their low performance. Nevertheless, this study suggests that if performance with mental computation is to be improved, much needs to be done in helping teachers formulate plans for addressing the area.

A second element of this study included examining student preferences for computation methods. Prior to taking the mental computation test, fifth-grade students were shown a number of multiplication problems and asked to indicate the computational alternative (calculator, mental computation, paper and pencil) that they preferred to use. A sample of the results from this preference survey is shown in table 4.3. Consistently, paper and pencil was the computational tool preferred by most students.

The researchers note several inconsistencies in the data. For example, they were surprised to find that 13 percent of the fifth graders chose to do 33 × 88 mentally. This problem does not seem to lend itself to mental computation, and it would have been interesting to learn what techniques these students would have employed. Further, a greater number of fifth graders preferred to compute 29 × 31 mentally than the seemingly easier item 36 × 25. This may reflect student misunderstandings of the task, which could be detected only by interview methods. Although table 4.3 does not show how successful students would be on these computations, it does indicate that they prefer alternatives other than mental computation.

Table 4.3
Fifth Graders' Preferences for Method of Computation

Item	Calculator	Paper and Pencil	Mental
36 × 25	9%	71%	20%
70 × 600	16%	45%	39%
29 × 31	7%	63%	29%
33 × 88	14%	73%	13%

As indicated earlier, all the published reports furnishing baseline data on students' performance with mental computation use a timed test and indicate that performance is low. Yet researchers who have interviewed students and allowed them to share their strategies for computing mentally indicate that at least some students are quite proficient at devising, using, and verbalizing some powerful strategies.

FUTURE DIRECTIONS

Curriculum and Instruction

If current recommendations for mathematics curricula are implemented, the elementary school mathematics classroom of the future will look quite different from that of the past or the present. A major, dramatic change in emphasis will occur from one driven by paper-and-pencil computation to one taking advantage of all the computational alternatives and emphasizing problem solving and mathematical connections. In addition to continued dialogue, research should be undertaken to learn the effects of elevating mental computation to the same status as other computational alternatives, such as the calculator, estimation, and written computation. Two groups of researchers in the United States are involved in ongoing research on the effects of a curriculum centered on student exploration and the communication of ideas—a curriculum free from direct instruction on written algorithms. These investigations are taking place in primary-grade classrooms, and each is exploring a type of instruction that is radically different from what is commonly used in most classrooms in the United States.

Cobb and his colleagues are studying the effect of a problem-centered second-grade curriculum that focuses on student interaction and self-generated solution methods (Cobb and Merkel 1989; Yackel, Cobb, Wood, Wheatley, and Merkel 1990; Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti, and Perlwitz 1991). Computation procedures are generated by students as they are needed in different problem situations. Often, these procedures are mental thinking strategies that build on prior knowledge. Students share solution methods in whole-class discussions after working in small groups. These researchers reported that all but one of the twenty students in the study had developed a variety of strategies to solve a range of problems. More than half were successful by mid-year in developing efficient methods for adding two-digit numbers. Although less conceptually advanced students tended to use less efficient methods (e.g., finger counting), the researchers argue that even these students were making progress by building on what is meaningful to them. The students in this project are also active and persistent in solving problems, and they expect their answers to make sense.

Kamii (1989) has followed primary-grade teachers in one school as they implement a program of mathematics that puts into effect Piaget's constructivist theory of learning. Classroom activities include using situations of daily living (e.g., daily lunch counts), group games (e.g., the card game "war"), and student discussion of problems. Student autonomy and construction of knowledge are principal elements of this program. Although standardized achievement data have not shown a difference between classes in the study and traditional classes, interview data with students in both groups document a difference in favor of the experimental group in student autonomy, in their conceptual understanding of place value, and in their ability to estimate and compute mentally.

The work of Cobb and his colleagues and of Kamii is promising and will likely furnish direction for future substantive curricular changes and teaching methodologies. Whether this effort and these results can be sustained will depend on several factors, including the extent to which teachers can adapt to this view of learning and mode of instruction.

Research exploring alternative approaches to instruction in mental computation and its effects on students is needed to guide future recommendations. This research should include investigations that explore the following:

- Direct and indirect methods of encouraging the development of self-generated thinking strategies
- How self-generated mental strategies interface with formally taught algorithms
- To what extent students of varying ability levels develop efficient mental strategies
- The relationship of the development of mental thinking strategies to general problem-solving development
- How self-generated mental thinking strategies apply to the study of non-whole number work (e.g., fractions, percent)
- The relationship between mental computation and estimation strategy development

Assessment

The methods used to evaluate student performance on mental computation include standard multiple-choice test items, open-ended test items orally or visually presented with or without a time restriction, and individual interviews. The latter method has been employed to document thought processes used to calculate mentally. It is ironic that whereas written computation is a major subtest of all standardized achievement tests in the United States, mental computation is conspicuously absent even though some exercises on these tests might be considered more appropriate for mental computation than for written computation (R. Reys 1985). However, the methods (written or mental) used by students to find solutions to standardized items on the computation subtest have not been identified. It is likely that the subtest may, in fact, be measuring skill in both written and mental computation.

In the future, it will be important to devise and validate a variety of evaluation methods so that baseline information on students' ability to generate and implement mental computation strategies can be collected. A student-interview format in which time is unrestricted will likely be useful for this purpose. This format will allow for the analysis of student thinking as well as for the effect of unlimited time on the generation of procedures.

Although standardized, large-scale instruments measuring mental computation performance may be difficult to generate and use, teachers can gather useful data on their own students' development of mental computation by employing a variety of in-class techniques. These include listening to students' responses as they verbalize strategies in a whole-class or small-group setting, or asking individual students to explain their thinking and choice of computation method in a one-to-one setting. Techniques and guidelines for assessing student progress in mental computation in a whole-class setting have been outlined by R. Reys (1985). They include varying the presentation format (visual and oral) of the problem, displaying visually presented items one at a time using an overhead projector, limiting the time per item, employing an open-ended response format, and limiting the number of items presented in a single session.

SUMMARY

In the United States today, mental computation is considered by a growing number of educators and researchers as the primary tool for small-number or for simple computations. Mental computation, in conjunction with a calculator for complex or for tedious computations, may replace to a great extent the need for paper-and-pencil algorithms.

A look in most classrooms today in the United States gives little cause to believe that self-generated thinking strategies for computation will replace or even be an important component of the curriculum or the instructional climate of the mathematics classroom in the foreseeable future. Nonetheless, we are optimistic. Our enthusiasm comes from the important work we see going on in a small number of classrooms where paper-and-pencil computation is receiving less emphasis and students' self-generated strategies for computation are valued. The effects of this atmosphere and emphasis on students' attitudes about mathematics and about themselves as learners and students' increased persistence in solving problems give cause for this optimism and continued commitment to effecting change. We are also enthusiastic because of the clear evidence that many students are capable of devising sophisticated strategies to compute and that these strategies have the potential for expanding and enriching their knowledge of mathematics at the conceptual level.

Although we are enthusiastic about the possibilities that mental computation holds for helping to redesign the current emphasis on computation at the elementary school level, we are also impressed by the need for both short- and long-term research efforts to address important questions. Although the time seems right for bringing about significant changes in the content and methodology of teaching computation, these changes will not occur without considerable teacher education, and this will take both time and a concerted effort by the entire mathematics education community.

Part 2

Similarities and Differences: Computational Estimation in Japan and the United States

Much has been written recently in curriculum guides and reports in both Japan and the United States about the importance of estimation. Even so, there has been surprisingly little implementation of estimation in either country's school mathematics programs. Although the history of support for estimation has been consistent throughout the twentieth century, that support has been accompanied by a general ambivalence about estimation: What is it? How is it developed? When and how should it be established? These are critical questions to resolve, but they must be addressed or they will continue to hinder future progress.

Language has contributed to some of the difficulties in both countries. Historically in the United States, estimation has many synonyms. For example, in some contexts, the terms *guess* and *estimate* mean the same, yet computational estimation involves far more than guessing. Estimation has also been synonymous with approximation and rounding numbers, which has allowed rounding techniques to dominate the instructional focus and obscure other, more important issues related to estimation. Similarly in Japan, language creates problems. For example, *gaisan* means "rough computation" and *gaisuu* means "rough number," and both have been used in Japanese mathematics programs. Yet *gaisan* and *gaisuu* are technical terms used mostly in school, whereas *mitsumori*, which literally means "see and expect," is most often used in Japanese social life. In an effort to expand estimation beyond rough computation and stimulate a fresh focus, the current movement in Japan uses the term *mitsumori*. This vocabulary change appears to be a stroke of genius, because it has created some controversy and stimulated greater interest in, and discussion of, estimation among Japanese teachers and mathematics educators. It has suggested a new image and encouraged a fresher look at estimation than has been generated thus far in the United States.

In both countries, there is the firm belief that estimation involves "higher-order thinking" and therefore must be an integral part of mathematics learning. Although this integration defies a clear characterization, the notion that estimation contributes to mathematical knowledge is clear. For example, the Japanese identify estimation as building the foundation for students to have a "good perspective" of numbers, whereas the Americans talk about estimation contributing to "number sense." Some Americans argue that number sense is a part of estimation, but others believe that estimation is a component of number sense. Although there are some differences of perspectives in both countries, there is agreement that estimation is important and that it has the potential to contribute significantly to students' mathematical thinking.

In the United States, estimation is viewed as a *process* that is involved in many facets of the mathematics curriculum. It is perceived as integrated and interacting with all phases of computation (such as reflecting on the reasonableness of a result) and as a process that grows and develops as students have regular experiences estimating. This broad view of estimation is reflected in the NCTM *Standards* documents but has yet to become widespread in school mathematics programs or in classroom instruction. The Japanese take a more pragmatic view of estimation, perceiving it as a *topic* in the mathematics curriculum. This view may eliminate some of the interconnections perceived by Americans. However, the task of handling estimation in curriculum development and instruction appears to be more straightforward when it is viewed as a specific skill linked to the acquisition of specific techniques that are useful in producing estimates.

The extent of instruction on estimation in both countries has been limited to an emphasis on rounding. Rounding, as a skill, has been taught in both countries, but these rounding procedures are typically rule driven, and connections to estimation are not always established. The idea that different estimates may be acceptable for the same problem initially tends to create confusion for students in both countries who have come to expect "one right answer" for a mathematics problem.

The attitude of many Japanese students toward computational estimation is negative. They believe that getting an exact answer is always necessary and is the most important part of mathematics learning. Although similar findings have been reported in the United States, research has also shown that the more students learn about estimation and develop estimation techniques, the greater their respect for estimation and tolerance for accepting a range of answers. Given the new emphasis on estimation in the Japanese curriculum, research designed to assess student attitudes toward estimation and monitor them over time seems both timely and appropriate.

The Japanese have high performance levels on mental and paper-and-pencil computation because students effectively use techniques that have been carefully and thoroughly taught. An obvious implication of this success would be to decide which estimation techniques are most effective and then to teach these techniques directly. Thus, there is considerable interest in Japan in identifying effective estimation techniques and strategies so they can be sequenced and systematically developed in school mathematics programs. Some powerful estimation strategies have been successfully taught to students in the United States, so there is research evidence to support this approach. However, American researchers support the idea of strategies that develop or evolve naturally from problem situations—students generating their own techniques and sharing them with the class—as a method preferable to direct instruction of strategies. Of course, there may be times when certain strategies need to be clarified or even introduced or demonstrated by teachers, but much of the responsibility for generating estimation strategies rests with students rather than teachers.

Both countries consider estimation to appropriately occur in real or problem-solving contexts. However, there seems to be more concern in the United States to involve students in some of the decision-making aspects of estimation: When is an estimate needed? Do I need an overestimate or an underestimate? How close is close enough? There is unanimous agreement that such questions need to be part of the mathematics curricula in the United States, even though it is acknowledged that they are not currently reflected in most school programs.

Estimation is directly assessed neither on most standardized tests in the United States nor on national or external exams in Japan. Assessment data of student performance in estimation in both countries is very limited, but the data that have been reported document low performance among both Japanese and American students. The assessment procedures typically used have raised serious questions regarding the validity of the data and have documented the need for nontraditional testing methods in Japan and the United States.

Helping teachers become aware of the changing emphasis on estimation presents challenges for both countries. Major adjustments need to be made in both in-service and preservice efforts to insure that the instructional attention being recommended is implemented.

Estimation is a common term long used in mathematics teaching in the United States, so teachers may feel that estimation is "old stuff" and simply not comprehend the scope and magnitude of the current movement. It is interesting that the Japanese word traditionally used for estimation is *gaisan* ("rough calculation"), but their new recommendation is to use a more formal term for estimation, *mitsumori* ("see and expect").

5

Computational Estimation: Curriculum and Instructional Issues from the Japanese Perspective

Shigeo Yoshikawa

UNTIL 1947, Japan had national textbooks of mathematics for elementary schools, which were prepared and published by the Ministry of Education. Every student learned mathematics through these national textbooks. Historically speaking, there were three types of the textbooks, which were named for their cover colors (Japanese Ministry of Education 1905, 1935, 1941):

- The black cover textbook (1905–1935)
- The green cover textbook (1935–1940)
- The blue cover textbook (1941–ca. 1947)

In the black cover textbook, the teaching of computation was systematized by making a connection among paper-and-pencil computation, mental computation, and the soroban (Japanese abacus).

The green cover textbook was called an epoch-making textbook because it had several new features. Naomichi Shiono, the main writer of the textbook, was famous for developing a good system for teaching computation. He also emphasized the importance of teaching mental computation (Shibata, chapter 2 in this volume). In the fifth graders' text, round numbers and estimation (rough calculation) were taught in one section, which also provided some opportunities for students to consider appropriate situations for estimating.

In the blue cover textbook, the basic philosophy on the teaching of computation was similar to the green cover textbook, and estimation problems were also included.

Since 1947, the teaching objectives and content in elementary and secondary school mathematics have been guided by the national curriculum, which is prepared by the Ministry of Education. Each curriculum was prepared by a committee in the Ministry of Education comprising school teachers, mathematics educators, and officers of the ministry.

Figure 5.1 shows the six curricula we have had since 1947.

Version	Year	Brief Description
1	1947	This first national curriculum aimed at developing students' mathematical concepts, their ability to think mathematically, and their scientific attitude.
2	1951	This curriculum emphasized the relationships between mathematics and students' daily-life problems.
3	1958	Since low performance was reported, the national curriculum was revised to recommend systematic teaching of mathematics.
4	1969	Systematic teaching was still emphasized in this curriculum. Some ideas of "new math" were introduced in this curriculum.
5	1977	Since some teaching materials related to new math were criticized for being too difficult for students, basic knowledge and skills of mathematics were the focus in this curriculum.
6	1989	In this latest version of the curriculum, the basic philosophy is that students should make good use of mathematical methods and should appreciate the values of mathematical ideas.

Fig. 5.1. Versions of the National Curriculum of Mathematics for Elementary Schools

In the first and second versions of the curriculum, attention was given to estimation. For example, in the introduction of the first version, the importance of mathematics education was stated, and the significance of the idea of estimation was discussed. Later, more specific descriptions of estimation were made. For example, "students should estimate the result of the problem and then they should compare it with the result after solving the problem."

Since the third version of the curriculum, the contents have been described simply and concisely, partly because the national curriculum has been determined by law since that time and all the textbooks published by commercial companies must be in accordance with it. Therefore the description of the curriculum must be in a simple and concise style. Estimation was not emphasized as much in the third, fourth, and fifth versions.

BACKGROUND OF THE PROBLEM

In the sixth version, more emphasis is given to the teaching and learning of estimation in elementary schools. Several factors have played a role in this revival of interest in estimation, some specific to estimation and others reflecting more general mathematics education concerns.

Students have much difficulty solving estimation problems. The Ministry of Education assessed and reported on student achievement in mathematics in 1984 (Japanese Ministry of Education 1984). The survey was conducted to get information on the effectiveness of the national mathematics curriculum. It reported that "generally speaking, students have good knowledge, they have good computational skills and they are good at applying basic formulas. But they have difficulty understanding mathematical thinking and making good use of it."

In this survey, one item was posed to the students to assess their performance on estimation (see fig. 5.2).

There is a number which was rounded to an integer after the calculation of 304.15 x 18.73. Do the rough calculation to find a correct answer, and choose a correct one.

1. 570 2. 5697 3. 56967 4. 569673

rough calculation:

answer:

Fig. 5.2. National assessment item on estimation (1984)

Although 62 percent of the students chose the correct answer, only 19 percent actually did rough calculation or estimation. The majority calculated an exact answer and then matched it to the closest answer. Many mathematics educators were surprised to learn that this was one of the most difficult items on the survey (Nakajima et al. 1984).

Other researchers assessed and analyzed students' performance and strategies on estimation problems (Yoshikawa 1986; Itoh et al. 1987, 1988). They reported similar low performance and found that students had much difficulty solving them.

Estimation is one of the basic skills in mathematics. During the last fifteen years, much effort has been devoted to research on the basics in mathematics. Estimation has been recognized as a basic mathematical skill for both everyday life and for learning mathematics (Kikuchi 1985; Kotou 1987).

Students should use the idea of estimation when learning mathematics. For example, estimation is useful in division problems like the following:

$$\begin{array}{rll} 44 & & \\ 18\overline{)792} & 18 \rightarrow 20 & 80/20 = 4 \\ \underline{72} & 79 \rightarrow 80 & \\ 72 & & \\ \underline{72} & & \\ 0 & & \end{array}$$

Estimation is important for problem solving. In 1989, the phrase *mitoushi o motsu* ("to have a good perspective") appeared in the *Curriculum of Mathematics for the Elementary School* for the first time. *Mitoushi o motsu* is often used by Japanese people in daily situations and has several meanings, including to "look through," to "forecast something," and to "expect what happens in the future." This phrase describes an objective of mathematics education in the Ministry of Education's new national curriculum. The use of this new term signals not only that new attention is being paid to estimation but also that estimation plays an important role when students solve mathematical problems. For example, estimation helps students have good perspectives when solving problems. When confronted with a problem, students should have some perspective both for how to solve it and for its result. Estimation has educational values that help contribute to the development of good perspectives (Yoshikawa 1987).

Another example of having a good perspective is the recognition of an error or an early awareness of an unreasonable answer. Students sometimes make a mistake in the process of computation:

$$\begin{array}{r} 508 \\ -\ 309 \\ \hline 299 \end{array}$$

It is good to have a perspective on the result of this computation before computing it. If students expect that the result is near 200, they soon realize that their result cannot be correct.

The teaching of estimation is related to the basic philosophy of the curriculum revision. National curricula have been prepared for all school subjects, including mathematics. All curricula are revised at the same time. Before the revision of the curricula, a special committee was formed in the Ministry of Education to discuss directions for the next curriculum revision. The committee reported the results of the discussion (Committee for the Discussion of Educational Curriculum 1987), in which the basic philosophy of the new national curricula was stated.

As for the mathematics curriculum, the committee suggested that mathematics education should respond to the changing society and that it should help develop students' logical reasoning and intuitive thinking.

More specifically, four basic directions were shown with respect to the revision of the elementary school mathematics curriculum. The first one was as follows: "Students should be able to grasp the rough magnitude of numbers and quantities as well as the rough shape of geometrical figures. Then they should be able to judge properly using their grasp of rough magnitude or rough shapes."

According to the official guide book, the curriculum of elementary mathematics was designed to help students have a good sense of numbers, quantities, and figures. Also, "students should become good estimators" (Japanese Ministry of Education 1989). The implementation of the new curriculum in elementary schools began in 1992.

Before closing this discussion, it is important to realize that there are many Japanese terms related to estimation. For example:

mitsumori	estimation
gaisan	estimation or rough computation
gaisuu	round number or rough number
gairyo	rough quantity
gaisoku	rough measurement
gaikei	rough shape or sketch
kinjichi	approximate value
kinji keisan	approximate computation

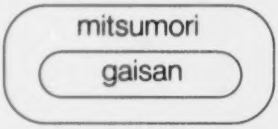
The two terms that are translated most closely into the English word *estimation* are *gaisan* and *mitsumori*. *Gaisan* literally means "rough computation." *Gai* means "rough," and *san* means "computation" or "counting." *Mitsumori* literally means "to see and expect." *Mi* means "see," and *tsumori* means "expectation" or "evaluation." There is also the term *gaisuu*. *Suu* means "number," so *gaisuu* means "rough number" or "rounded number."

The two terms *gaisan* and *gaisuu* have been used in mathematics for a long time. These terms were used even in the green cover textbook published in 1935. In more recent mathematics textbooks, we also find the terms *gaisan* and *gaisuu* used often. These terms are familiar to students as well as to mathematics teachers. An examination of textbooks and teaching shows that the main focus is on *gaisuu*, the skill of making rounded numbers. *Gaisan* ("rough computation") is introduced as merely one of the applications of *gaisuu*.

According to the previous national curriculum of mathematics released in 1977, *gaisuu* ("rounded number") is to be taught in the fourth grade. The curriculum states, "Students should understand how to make a rounded number and be able to use it for a purpose. Students should know *sisyagonyu* ['straight rounding']." In the reference book published by the Ministry of Education in 1986, one of the chapters is entitled "Rounded Number and Calculators" (Japanese Ministry of Education 1986). *Gaisan* ("rough computation") is not emphasized in this chapter. This tendency can be seen in many textbooks used before the new curriculum was implemented in 1992. Our impression is that in the actual classroom teaching, teachers were emphasizing the meaning of *sisya-gonyu* ("straight rounding") and trying to develop students' skill in *sisya-gonyu*. Thus, *gaisan* was not viewed as a very important topic in the classroom.

We began to use the term *mitsumori* in our educational discussions rather recently. *Mitsumori* is used in the current *Curriculum of Mathematics for the Elementary School* and its importance is explained. But we have introduced a controversial issue with respect to the definition of estimation: should we recognize the difference between *gaisan* ("rough computation") and *mitsumori* ("estimation")? Some mathematics educators say there is no difference between these two terms; others insist that we should identify *mitsumori* with *gaisan*.

I think *gaisan* is a simple skill, whereas *mitsumori* contains highly ordered mathematical thinking. As this figure shows, *gaisan* should be a subset of *mitsumori*, which includes many thinking activities involving numerical evaluation or judgment in mathematical situations (Yoshikawa 1986).



WHAT IS HAPPENING?

Trends in teaching estimation. The *Curriculum of Mathematics for the Elementary School* is divided into three parts:

1. Objectives of elementary school mathematics
2. Content and treatment for each grade level
    a) Number and computation
    b) Quantity and measurement
    c) Geometrical figures
    d) Numerical relations
3. Remarks on making teaching plans

In the fifth version of the curriculum, the objectives were stated as follows: To acquire the basic knowledge and skills concerning numbers, quantities, and geometrical figures and to develop abilities and attitudes to understand daily-life problems mathematically, to think about them logically, and to solve them.

In the sixth (current) version, the objectives are as follows: To acquire the basic knowledge and skills concerning numbers, quantities, and geometrical figures; to develop abilities to think about daily-life problems logically and to have good perspectives; and to develop attitudes to appreciate the value of mathematical thinking and to make good use of mathematical methods in everyday life.

These statements have connections to teaching estimation. Estimation will help students have "good perspectives" when solving problems.

Students should "appreciate the value of mathematical thinking," including the idea of estimation, and they should "make good use of mathematical methods," such as using estimation.

The content concerning estimation is stated in the second section. Addition and subtraction of two- and three-digit numbers are to be taught in the second grade. According to the current version of the curriculum, teachers should help students consider their method of computation and check the result of computation by using the idea of estimation.

Estimation is to be taught in the fourth grade. The following is a part of the contents included in "Number and Computation": "To understand how to round numbers, to be able to make use of rounded numbers for some purpose, and to appreciate the value of rounded numbers." This curriculum also says that teachers should help students recognize situations where round numbers are used; understand the meaning of *sisya-gonyu;* represent the result of computation using round numbers for some purpose; and estimate a sum or difference using rounded numbers.

In the fifth grade, students are to develop the ability to make use of rounded numbers. For example, they should learn to estimate a product or quotient for some purpose. According to the third section ("Remarks on Making Teaching Plans"), teachers should "help students make reasonable judgments or invent effective methods by using estimation of numbers and geometrical figures."

Curricular Issues

Here—in my opinion—are some problems concerning the treatment of estimation as recommended in the current national curriculum.

Relationship between rounded numbers and estimation. Although rounded numbers and estimation are connected with each other in the curriculum and have a close relationship, estimation is not always subordinate to rounded numbers. Estimation is also closely related to the mathematical structure of the problem and the characteristics of numbers.

> *Example 1. Taro wants to buy two books. The prices are 245 yen and 815 yen. He has 2000 yen. Can he buy these two books?*
>
> To solve this problem, it is helpful to round the numbers: 245 → 250 and 815 → 800. The sum of 250 and 800 is 1050, which is smaller than 2000. Thus he can buy the two books. However, it is essential to derive an observation like "Both numbers are smaller than 1000, so the sum is smaller than 2000" from this problem.
>
> *Example 2. Estimate 125/0.24.*
>
> Since 0.24 is close to 1/4, we can translate this problem into 125 × 4, and so 500 is an estimate. This method is related to the mathematical structure of the problem.

The national curriculum does not explain very minutely the content to be taught. As mentioned earlier, the description of the curriculum is rather concise. Many pragmatic issues are not addressed. Estimation problems vary from very easy to very difficult, but the degree of difficulty is not discussed, nor are there guidelines on which estimation skills are appropriate to teach to students.

The treatment of estimation in textbooks. Generally speaking, authors of textbooks do not want to make their textbooks very difficult. Therefore, they might avoid introducing many estimation problems. Since estimation has received limited attention in the textbooks in the last thirty years, authors might hesitate to prepare or develop situations and materials for teaching estimation.

Instructional Issues

Concerning teachers. Many teachers value highly the ability to compute with paper and pencil. Teachers and parents believe that one's mathematical ability is equal to one's ability with paper-and-pencil computation. Teachers generally tend to avoid introducing hand-held calculators into their mathematics classrooms because they are afraid that calculators would prevent students from developing the ability to compute using paper and pencil. The probability is high that teachers would avoid introducing estimation problems into their classrooms for the very same reason.

Some teachers do not like the ambiguity included in estimation problems. These teachers like to teach how to round a number because the process is rather simple and clear.

Concerning students. Students have a limited view of mathematics. They think that every mathematical problem allows only one approach and has only one solution. Since students tend to stick to the approach that was taught before in the classroom, it is difficult to develop and apply new methods. Because students have learned a lot about paper-and-pencil computation in mathematics classrooms, they tend to use paper-and-pencil methods for whatever computational problem is posed.

Yoshikawa (1986) reported that students have difficulty using the idea of estimation when solving problems. The problem in figure 5.3 was posed to 79 fourth graders and 80 fifth graders. The results were as follows:

Fourth graders ($n = 79$)

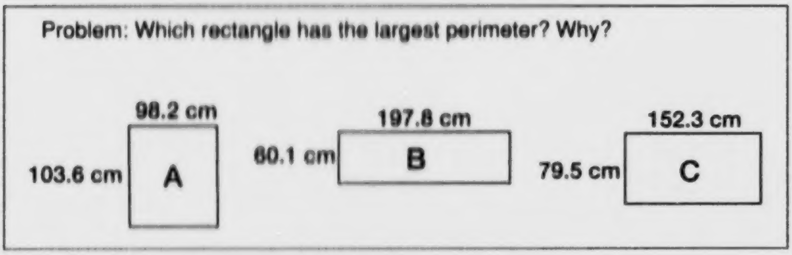
Using exact computation	44 students
Using estimation	12 students
No reason	2 students
Incorrect answer	21 students

Fifth graders ($n = 80$)

Using exact computation	70 students
Using estimation	0 students
Incorrect answer	10 students

Although this problem could have been solved by estimation, very few students estimated. After students solved this problem, some were selected to be interviewed. Half of them were the students who estimated when solving this problem, and the other half were those who did not. Students' activities and attitudes were summarized as follows:

- Most students mentioned they usually did not use estimation in the mathematics classroom. Although they had learned some methods for estimation, they tended to avoid using it.
- The main reason why students decided to use estimation was that after they tried to do exact computation, they felt it was complicated or tiresome for them.
- Some students mentioned they were using estimation in daily-life situations, but they never tried to use it on formal tests conducted in mathematics classrooms.
- When the interviewer presented the idea of using estimation to the students who did not estimate, some of them appreciated the value of it. Others said they did not like it. One student said she was confident with paper-and-pencil computation, so she wanted to use it all the time. She also said, "I don't like estimation. I think it is an evil method." She seemed to feel that using estimation was "immoral."



Problem: Which rectangle has the largest perimeter? Why?

Fig. 5.3

WHAT SHOULD HAPPEN?

Several recommendations concerning future directions for estimation follow.

Make good connections between estimation and other computation methods. Several methods of computation are given in the mathematics curriculum:

paper-and-pencil computation, mental computation, estimation, and the calculator. We should make good connections among these four methods in the curriculum.

Japanese mathematics educators have tried to develop a good system for teaching paper-and-pencil computation ever since the publication of the green cover textbook. The degree of completeness of the system is rather high. It is one of the good traditions in Japanese mathematics education. Teaching mental computation also has been a tradition. However, estimation has received limited attention in the curriculum and in the classroom. We should try to establish good links between estimation and paper-and-pencil computation and also between estimation and mental computation. They compensate each other.

It is important to make good use of the idea of estimation when doing computation with written algorithms. For example, some students have difficulty when adding two fractions and make an error like $1/2 + 2/3 = (1 + 2) / (2 + 3) = 3/5$. Using the idea of estimation, students know that the sum of the two fractions $(1/2 + 2/3)$ must be larger than 1 and thus notice that their result (3/5) is not reasonable. They will be able to do paper-and-pencil computation in a confident manner when they develop the ability to check the process and the result of the computation by using estimation. It is reported that Japanese students lack the conceptual understanding of fractions and tend to rely on a mechanical and algorithmic approach when computing with fractions (Reys et al. 1991). We should improve this situation.

Mental computation is deeply related to estimation. We should develop our curriculum from the viewpoint that students should get enough power from mental computation to allow them to estimate. This viewpoint will help us determine the scope and sequence of teaching mental computation.

Some students feel uneasy when doing computation or when making some judgment based on estimation. It is good for the teacher to recommend that they use a calculator to check their computational results or their judgment. In another situation, students could check the result of their calculator by using estimation.

Help students develop good perspectives. Students should develop two kinds of perspectives when solving mathematical problems: perspectives for the approach, and perspectives for the result. Using estimation is an effective way to develop good perspectives (Yoshikawa 1987, 1989).

Example 1. Perspectives for the approach. When students confront a word problem that includes some complicated or "messy" numbers, they often have difficulty choosing the appropriate operation to solve it. Estimating with these numbers can help students choose the correct operation. This is a kind of thought experiment.

Example 2. Perspectives for the result. If a problem like "Find the average of 3, 8, 9, and 10" is posed, students should notice that the average of the four numbers must be between 3 and 10. Some students might anticipate that the average is "closer to 10 and not close to 3." This kind of estimation helps students not to make a big mistake.

Prepare suitable situations for estimation. Sometimes estimation should be taught as a single topic. Estimation skills have much educational value. Teachers should encourage students to apply them when solving problems. In principle, however, estimation should be taught in context—that is, in problematic situations. One reason is that it is easier for students to appreciate the value of estimation when it is used in the right situations; they realize that using estimation can save effort or energy.

A second reason is that students should develop the ability to find several different approaches to one problem and then choose the most appropriate one. More specifically, students must judge whether using estimation is reasonable in a certain context.

A third reason is that the degree of exactness of estimation usually depends on the situation or the context. Some students mention that they feel uneasy when estimating because they are not sure how precisely they should estimate.

The lesson plan shown in figure 5.4 reflects the remarks mentioned above. In this lesson, students can discuss their approaches by examining and comparing them. Through the discussion they learn to value estimation, and they can become confident in using estimation.

Broaden students' and teachers' view of mathematics. It is reported that students tend to have a limited and fixed view of mathematics. For example, they think that every mathematical problem has only one approach (Nohda 1983; Yoshikawa 1990). Students believe that paper-and-pencil computation is always the most important method. Therefore, they do not stop to think which computation method is appropriate and suitable in a problematic situation posed to them. These beliefs will hinder the teaching and learning of estimation. Teachers should help students broaden their view of computational alternatives.

At the same time, teachers should also modify their own fixed viewpoints. In order to improve the situation, we should prepare the new curriculum for the preservice teacher. In this new curriculum, the teaching and learning of estimation should be settled enough to broaden preservice teachers' views and to help prepare them for teaching estimation. Teachers and students alike should realize that estimation is an essential part of mathematics.

Looking back over the history of mathematics education in Japan, we see that teaching estimation was emphasized to some degree at some

Grade 4 lesson plan Topic: Round numbers and estimation

Objectives:
1. To introduce round numbers
2. To help students make good use of estimation in problem solving

Development of the lesson

Teacher's activities	Student Activities	Remarks
Introduce round numbers.	Give some examples of exact numbers and round numbers.	Divide numbers into two groups.
Teach some terms; pose a problem.	About, around, round number Problem: Taro is going to buy three things in a grocery store. The prices are 412 yen, 325 yen, and 541 yen. He has 1000 yen. Can he buy all these things?	Check if all students understand the meaning of the problem.
Help students solve the problem.	(Expected approaches) 1. Using exact computation 2. Using rough computation (estimation)	Encourage students to use their favorite approach.
Make students discuss the approaches of problem solving.	Explain one's own approach. Discuss advantages and disadvantages of their approaches: estimation vs. exact computation.	Admit that every approach has some advantages.
Summarize today's teaching and learning.		Pay attention to the value of mathematical thinking.

Fig. 5.4. Lesson plan

time, but the emphasis did not last for a long time. It is high time that we build a stable base for teaching estimation in mathematics classrooms. We should also clarify the educational values of estimation and its place in mathematics curricula to mathematics educators, teachers, and parents.

We have been making a rigid system of teaching paper-and-pencil computation for more than fifty years and are still trying to improve it. We must modify the teaching system in order to respond to the new society and the new century in which our children are to live. One of the innovations of our new curriculum is to introduce a new perspective of *mitsumori* ("estimation"). We must recognize that *mitsumori* is different from the skill of doing rough computation.

6

Assessment of Computational Estimation

Setsuro Itoh
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CALCULATORS are found in nearly every home and school, and when any computations are necessary, most people use a calculator. In offices, computers are available for more complex computations. But in spite of these situations, the attention given to paper-and-pencil computations in arithmetic in elementary school classrooms has changed very little this century. Table 6.1 presents a brief scope and sequence of arithmetic in Japanese elementary schools. It illustrates some of the elements of the "Number and Calculation" domain in our curriculum.

Table 6.2 estimates the percentages of time devoted to the "Numbers and Calculations" domain in our current elementary mathematics curriculum. It shows that much time (over half in grades 1–4 and more than one-third of the total instructional time in grades 1–6) is devoted to computation.

The computational stranglehold on the elementary school mathematics curriculum is breaking. More and more problem solving is being used to furnish contexts for students to solve problems and also to carry out some of the computations. Although calculators could be used to do the computation, they are rarely used in school (Nagasaki 1987a; also see Senuma's chap. 11 in this monograph.) As the emphasis on written computation shifts, there is a growing recognition of the importance of estimation. Unfortunately, we do not have a curriculum in which children's abilities in computational estimation can grow, nor do we have appropriate methods of assessment.

WHAT HAS HAPPENED?

Computational estimation has received very little attention in the Japanese elementary school mathematics curriculum, but rounding is taught. For example, in the previous *Curriculum of Mathematics for the Elementary School* (Japanese Ministry of Education 1977), students at grade 4 learned—

- to understand a round number and to use it according to the purposes;
- to know the meaning of rounding to a nearest whole number.

Table 6.1
Arithmetic Content at Each Grade Level

Grade	Numbers	Calculations
1	• Up to 2-digit numbers	• Addition and subtraction of 2-digit numbers
2	• Up to 4-digit numbers	• Addition and subtraction of 3-digit numbers (paper-and-pencil computation) • Multiplication of 1-digit numbers
3	• Up to 8-digit numbers • Simple decimals • Simple fractions	• Meanings of addition and subtraction of decimals and fractions • Multiplication of 3-digit numbers by 2-digit numbers (paper-and-pencil computation) • Division by 1-digit divisor (paper-and-pencil computation) and meaning of remainder
4	• Up to 16-digit numbers • Round numbers • Decimals and fractions	• Multiplication of numbers larger than 3-digit numbers • Division by 2-digit divisor (paper-and-pencil computation) • Addition and subtraction of decimals and fractions with common denominator • Multiplication and division of decimals by integers
5	• Odd and even numbers • Divisors and multiples	• Multiplication and division of decimals • Addition and subtraction of fractions • Multiplication and division of fractions by integers • Estimation of products or quotients
6	• Relationship of integers, decimals, and fractions	• Multiplication and division of fractions

Table 6.2
Rough Percentages of Instruction Time Devoted to Numbers and Calculations

Grade	1	2	3	4	5	6	Total
Numbers	34	22	14	21	22	6	19
Calculations	45	53	53	30	21	18	36
Total	79	75	67	51	43	24	55

And at grade 5, they learned—

- to develop the abilities to use round numbers through estimating products or quotients and through expressing the results of computation by round numbers. For example, the former means a student would estimate 471 × 905 as 500 × 900 = 450 000, and in the latter a student would compute 471 × 905 as 426 255 and explain that this answer is about 430 000.

Although students know how to round a number, they do not typically apply this skill when asked to estimate. Research shows that rather than estimate, a majority of Japanese students compute an exact answer

(Japanese Ministry of Education 1984). In the previous chapter, Yoshikawa has illustrated this phenomenon.

Before rounding is taught, children must rely on their number sense. Children (grades 1–3) have difficulty using rough sizes of numbers and believe they must give an exact answer whenever they compute. When asked to estimate, most students hesitated, but a majority responded correctly to question 1 below (Itoh et al. 1987, 1988). About one-fourth of the second graders may have used *kiriage* ("rounding down") and rounded 29 to 20, and about the same number of second graders responded with an exact answer (see table 6.3).

There are 52 boys and 29 girls at the park.

Q1: About how many boys (to the nearest ten) are there?

Q2: About how many girls (to the nearest ten) are there?

Table 6.3
Percent of Students Giving Each Answer

Q1:	Response	50	52	60	51	Others	Nothing
	Grade 2	57	24	5	1	14	—
	Grade 3	85	7	2	0	5	1
Q2:	Response	30	20	29		Others	Nothing
	Grade 2	39	23	25		12	1
	Grade 3	73	14	7		4	2

Note: *N* (Grade 2) = 250, *N* (Grade 3) = 250

In spite of the wording "About how many ... (to the nearest ten)," these students may not understand the meaning of *about* and *to the nearest ten,* or perhaps they believe that an exact number must be used in arithmetic classrooms.

Even though rounding is not taught until fourth grade, the third graders showed a dramatic rise in their use of rounded numbers. A majority (73% to 85%) may have used their number sense to give good estimates. Fourth graders are taught to round numbers, and this experience produced increasingly higher performance on problems involving rounding. For example:

The populations of children in elementary schools at a city are—

North town	*2336*	*South town*	*6547*
East town	*8762*	*West town*	*4109*

Q3: About how many children (to the nearest hundred) are there at North town?

Q4: About how many thousands and hundreds of children are there at South town?

Table 6.4 gives the responses.

Table 6.4
Percent of Students Giving Each Answer

Q3:	Response	2330	2300	2000	3000	2336	Others	Nothing
	Grade 3	5	69	5	0	9	17	4
	Grade 4	0	81	2	0	5	11	1
	Grade 5	0	89	1	0	3	6	1
	Grade 6	0	90	2	0	3	5	1
Q4:	Response	6550	6500	7000	6000	6547	Others	Nothing
	Grade 3	5	58	4	2	10	18	5
	Grade 4	3	74	5	1	4	12	1
	Grade 5	2	79	3	0	3	13	0
	Grade 6	1	81	1	—	4	12	1

From 58 percent to 69 percent of the third graders correctly answered the questions involving about how many thousands and hundreds. Only data on *kiriage* rounding situations were collected, so the student's ability to answer similar questions, requiring other rounding (such as the East town, which would use rounding up, or *kirisage*) is uncertain. As compared with third graders, children at grade 4 used round numbers appropriately, and their rates of correct answers increased about 20 percent. Modest improvement continued from grade 4 to grade 6, but the dramatic change in grade 4 clearly documented the impact of teaching rounding. The students were also asked these two questions related to estimation:

> *Q5: In thousands and hundreds, what is the difference between the populations of North and South town?*
>
> *Q6: How many ten-thousands and thousands of the population are in all the cities?*

Table 6.5 shows that a majority of students in grade 4 and above correctly used *kiriage* to report an estimate of 4200, with about three-fourths of the sixth graders successfully using *kiriage*.

The question requiring an estimate for all four populations was more difficult for all grades. Table 6.5 reports that a majority of fourth graders and three-fourths of the sixth graders produced estimates ranging from 20 000 to 22 000. However, the extent to which computing exact answers and rounding contributed to these results is not clear since many mistakes in the addition of these 4-digit numbers were observed.

Computational estimation was assessed in earlier grades by asking students to do computation and determining which numbers they would use to estimate. For example, second graders were asked to do the following:

81 − 38

Q7: ( ) − ( ) ... use an expression with rough sizes of numbers.

Q8: The difference is about ( ).

Table 6.5
Percent of Students Giving Each Answer

Q5:	Response	4 210	4 200	5 000	4 000	4 211		Others	Nothing
	Grade 3	5	44	3	7	7		29	6
	Grade 4	5	55	2	8	3		27	1
	Grade 5	2	72	1	2	2		21	1
	Grade 6	0	74	—	3	5		17	1
Q6:	Response	22 000	20 000	21 000	21 600	21 700	21 754	Others	Nothing
	Grade 4	26	3	16	5	4	5	40	2
	Grade 5	45	2	13	1	4	3	31	2
	Grade 6	50	2	11	2	5	3	25	2

Table 6.6 shows the students' responses.

Table 6.6
Percent of Students Giving Each Expression and Answer

Q7:	Response	80 – 40		80 – 30		81 – 38	Others	Nothing
	Grade 2	30		18		20	25	7
Q8:	Response	40	50	43	42	41	Others	Nothing
	Grade 2	28	15	20	4	1	28	3

Note: N = 250

The results in table 6.6 show that about half of the second graders transformed the problem into easier numbers with which to compute, and almost all of them were able to perform the computation correctly.

A similar problem was given to third graders and fourth graders:

5183 + 2178

Q9: ( ) + ( )... use an expression with rough sizes of numbers.

Q10: The sum is about ( ).

Table 6.7 gives their responses.

Table 6.7
Percent of Students Giving Each Expression and Answer

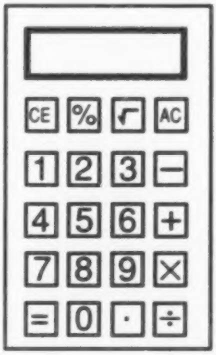
Q9:	Response	5000 + 2000	5200 + 2200	5183 + 2178	5000 + 2200	5100 + 2100	5000 + 2100	Others	Nothing
	Grade 3	15	36	8	—	10	0	25	5
	Grade 4	14	48	2	0	15	0	18	3
Q10:	Response	7000	7400	7361	7200	7100		Others	Nothing
	Grade 3	19	32	2	11	0		32	4
	Grade 4	15	43	3	16	1		21	1

About one-third of the third graders and almost half of the fourth graders correctly followed the directions to produce their estimate. About 15 percent of the students in each grade used *kiriage* and responded "5000 + 2000," and about 10 percent responded "5100 + 2100." Although they all did not provide the precision for the estimate suggested, about two-thirds of the students produced a reasonable estimate and fewer than 10 percent reported an exact answer.

Some research has also been directed toward determining the students' choice of suitable strategies and techniques, in addition to producing estimates. For example, in question 11, item 3 was considered to be the most suitable choice because this answer posed a critical check of the result based on the concept of the number sizes. Although the suitability depends on the problem situation and its mathematical structure, this type of research question is very important in promoting students' abilities with numbers and operations. Several questions were designed to learn about strategies students choose. The following question involves checking the reasonableness of a calculator-produced result.

Examine the answer shown on a calculator that expresses the result of the computation 637 × 8.

Q11: Select the idea nearest yours from the following:



(1) The answer 5696 is right.

(2) 600 × 8 = 4800. The answer 5696 may be wrong because 5696 is larger than 4800.

(3) 700 × 8 = 5600. The answer 5696 is wrong because 5696 is larger than 5600.

(4) The answer is wrong, because it is not equal to the result from a paper-and-pencil computation.

(5) I don't know.

Table 6.8 gives the results.

Table 6.8

Percent of Students Giving Each Choice

Q11:	Response	(1)	(2)	(3)	(4)	(5)	Nothing
	Grade 3	8	26	14	46	4	2
	Grade 4	4	16	11	64	4	1
	Grade 5	4	26	25	43	2	0
	Grade 6	5	18	33	41	2	1

Between 40 percent and 60 percent of the students chose not to use estimation strategies and instead wanted to compare the result to an exact answer. Although (3) is the best choice, only 11 percent to 14 percent of the third or fourth graders and 25 percent to 33 percent of the fifth or sixth graders selected it.

Following is a question designed to determine what numbers students used to produce their estimate:

> *If 496 sunflower seeds are divided by 68 pupils equally, then about how many seeds does each pupil receive?*

Q12: Which of the following computations would you use to get the answer?

(1) 496 ÷ 68
(2) 496 ÷ 70
(3) 490 ÷ 68
(4) 500 ÷ 68
(5) 490 ÷ 70
(6) 500 ÷ 70
(7) Others … [ ]

Table 6.9 shows students' responses.

Table 6.9
Percent of Students Giving Each Choice

Q12:	Response	(1)	(2)	(3)	(4)	(5)	(6)	(7)	Nothing
	Grade 4	29	5	8	7	11	40	1	—
	Grade 5	53	1	2	5	8	30	1	1
	Grade 6	41	3	2	4	10	31	0	1

About 30 to 40 percent of the students in grades 4, 5, and 6 selected (6), straight rounding 500 ÷ 70, whereas only about 10 percent of them selected (5), compatible rounding 490 ÷ 70. It should also be noted that the rest of the students used the "raw number" 496 or 68, or both. This tendency revealed that a difficulty of estimating existed in bridging from one exact number to a rounded number.

In an effort to gain a better perspective of students' number sense and also to learn how they estimate with fractions, fifth graders were asked the following questions. Table 6.10 shows the results.

Q13: Select one of the choices that may be a correct answer to this computation: 7/12 + 5/9

(1) It is larger than 1.
(2) It is smaller than 1.
(3) I don't know.

Q14: Select one of the choices that is nearest to the idea you used in response to the question above.

(1) 7/12 + 5/9 = 12/21, so it is smaller than 1.
(2) Reducing to a common denominator and computing it, 7/12 + 5/9 = 21/36 + 20/36 = 41/36, so it is larger than 1.
(3) 7/12 + 5/12 = 1, and 5/9 > 5/12 , so it is larger than 1.
(4) Both 7/12 and 5/9 are larger than 1/2, so the sum is larger than 1.

Table 6.10
Percent of Students Giving Each Choice

Q13:	Response	(1)	(2)	(3)		Nothing
	Grade 5	92	6	1		1
Q14:	Response	(1)	(2)	(3)	(4)	Nothing
	Grade 5	13	77	4	5	2

Although over 90 percent of the students correctly reported the sum to be greater than 1, most students (77%) decided to compute an exact answer to defend their choice. Their choice is probably a reflection of the fact that they have not been taught to estimate fractions. The values of the fractions involved (each greater than 1/2) lend themselves to choice (4), but very few of the fifth graders used that rationale.

The following question involving decimal computation was asked:

0.781 × 0.64

Which of the following is the correct answer?

(1) 0.0499
(2) 0.499
(3) 4.99
(4) 49.9

Q15: Write the expression with rough numbers. ( )
Q16: Which of (1) to (4) is nearest to the correct answer?

Table 6.11 shows the results.

Table 6.11
Percent of Students Giving Answers and Choices

Q15:	Response	0.781 × 0.64	0.78 × 0.6	0.8 × 0.6	0.78 × 0.6	Others	Nothing
	Grade 5	5	11	41	12	32	7
	Grade 6	4	11	53	12	16	4
Q16:	Response	(1)	(2)	(3)	(4)		Nothing
	Grade 5	15	62	15	8		1
	Grade 6	9	65	20	6		1

About half of the students used a good estimation strategy (0.8 × 0.6) for finding the order of magnitude of 0.781 × 0.64.

Following is a question given to sixth graders:

29/30 × 12/23.

Hanako says that the answer is about 1/2.

Q17: Guess her reasoning and then write the expression with rough fractions. ( )

Table 6.12
Percent of Students Giving the Rough Fractions Shown

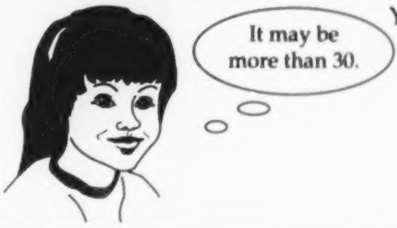
Q17:	Response	1 × 1/2 = 1/2	29/15 × 6/23 = 174/345 ≈ 1/2	Others	Nothing
	Grade 6	55	13	19	13

Fifty-five percent of sixth graders used 1 × 1/2 to confirm Hanako's estimate. The only other frequently used approach (used by 13%) was the more tedious process 29/15 × 6/23 = 174/345 ≈ 1/2. Students probably use this process because they have already learned to multiply fractions, produce exact answers, and when necessary reduce those answers.

WHAT IS HAPPENING IN THE CURRICULUM?

The learning of computational estimation has been increased and strengthened in the current national curriculum (Japanese Ministry of Education 1989) as compared with our old curriculum, but the changes in the new curriculum are still not sufficient to allow children's abilities in computational estimation to grow. Although the content in table 6.1 will be about the same, computational estimation will be taught in the same grade level as the exact computation. Table 6.1 gives a brief overview of numbers and computations, and the current curriculum proposes that estimation be taught beginning in the second grade. The previous curriculum included only *gaisuu* and *gaisan* in fourth and fifth grade, whereas the latest curriculum opens the door to *mitsumori* from second grade and after, whenever the content lends itself to *mitsumori*.

We will teach computational estimation along with the four arithmetical algorithms. Opportunities will be provided to help students appreciate estimation and help them solve mathematical problems insightfully by using estimation. Here are a few examples of estimation now found in Japanese textbooks.


It may be more than 30.

Yoshiko picked 24 flowers and Akira picked 13 flowers. How many flowers did they pick? Write an expression.

24 + 13

(Grade 2)

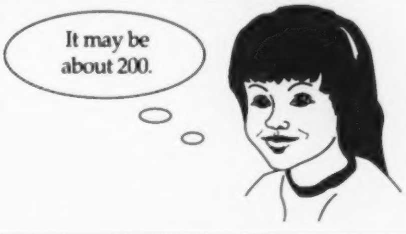
Before children get an exact answer by computation, they estimate that the sum may be more than 30 through the following process:

$$24 + 13 = (20 + 4) + (10 + 3) > 20 + 10 = 30 \text{ (by mental computation).}$$

There are 479 students in the school where Akiko goes. If there 234 boys, then how many girls are there?

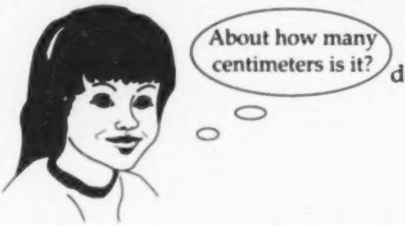
479 – 234

(Grade 2)


It may be about 200.

Second graders can use a front-end strategy for estimating the result of a subtraction: (3 digits) – (3 digits). In this example,

$$479 - 234 = 400 - 200 = 200 \text{ (by mental computation).}$$


About how many centimeters is it?

We used 4 pieces of 16-cm ribbon to make a flower decoration. Then how many centimeters of the ribbon did we use in all?

16×4

(Grade 3)

This problem will be useful in encouraging students to determine bounds for the result. For example, they may estimate a range of the product as follows:

$$10 < 16 < 20 = 10 \times 4 < 16 \times 4 < 20 \times 4$$

4 pupils divide 256 pieces of colored paper equally. Then, how many pieces does each one have?

$256 \div 4$

(Grade 3)

About how many pieces is it?

Children should be able to estimate that the answer is a little larger than 60 because $256 \div 4 > 240 \div 4$. Therefore, when they do a paper-and-pencil computation, they find that the tens place of the quotient is 6, so the answer is at least 60.

About how much is it?

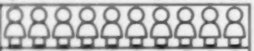
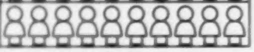
126 pupils of grade 4 visit the botanical gardens, and the expense of each one amounts to 365 yen. How much is the total expense?

(Grade 4)

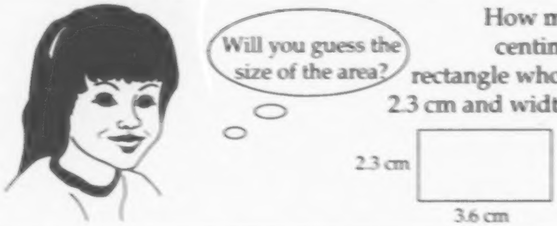
Find the rough numbers that are rounded to the 1/10 place in the quotients in the following divisions.

$7\overline{)24.3}$ $82\overline{)117}$ $47\overline{)96.6}$

Express the populations of Higashiyama-city and Kitagawa-city by picture graphs.
(1) How many ten-thousands are in the population of Higashiyama-city?
(2) How many ten-thousands are in the population of Kitagawa-city?

Higashiyama-city	68 418 人	
Kitagawa-city	81 756 人	

(● =10 000 人)


Will you guess the size of the area?
2.3 cm
3.6 cm

How many square centimeters is the rectangle whose length is 2.3 cm and width is 3.6 cm?

(Grade 5)

Find the rough numbers that are rounded to the upper two figures of the quotients in the following divisions.

$0.28\overline{)1.62}$ $0.48\overline{)0.4}$ $0.58\overline{)4.638}$

If we can paint 1 3/4 m^2 of a board with 1 dL of paint, then how many square meters of the board can we paint with 2 1/3 dL of paint?

1 3/4 × 2 1/3

(Grade 6)

Will you guess the answer?

Problems such as these appear in different grade levels so that Japanese students will have opportunities to make estimates much more frequently.

FUTURE TRENDS

Mathematics is changing, and so are the needs of Japanese children. Schools must use less time and labor in the future to develop paper-and-pencil computations in arithmetic. This does not mean that all paper-and-pencil computations should be eliminated, but less instructional time should be devoted to these skills. This change would allow more time for teaching estimation and mental computation and using calculators more wisely. This change in emphasis would also yield opportunities to teach more mathematical thinking and help students develop "good perspectives" of mathematics.

Teaching estimation may actually promote the more efficient development of paper-and-pencil computations. Estimation encourages students to think about numbers and ways to handle them easily. Many questions about teaching estimation need to be considered. When should estimation be taught? Should it be taught before exact computation or after? Because the procedures for estimation are fundamentally different from those for paper-and-pencil computations, children may become confused if they are taught both at the same time. Does the teaching of estimation promote better conceptual development? Does estimation help mathematical problem solving? How can we help students perceive estimation as something more than just a way to check the accuracy of their computation? What can be done to help students recognize and appreciate estimation as an important part of mathematical learning? How should testing be revised to accommodate this changing emphasis on written computation and estimation? How can valid measures of estimation be obtained? Thoughtful discussions of these and many other questions need to take place to guide future change in our schools.

7

Computational Estimation: Curriculum and Development Efforts and Instructional Issues

Paul R. Trafton

THROUGHOUT this century, curriculum reports in the United States have repeatedly supported a major emphasis on computational estimation. The reasons are many: it aids in developing quantitative thinking, it's a good strategy for solving word problems, it's a useful check for paper-and-pencil computation, it's a valuable tool for guiding and monitoring exact computation, and it is practical to use in everyday life. Despite these reports and reasons, little attention has been given to the systematic development of estimation in curriculum materials, and little research has been devoted to its acquisition by students or to the effects of estimation instruction (Benton 1986).

Evidence of low achievement on estimation tasks is consistent (Carpenter et al. 1984; National Assessment of Educational Progress 1983; Reys et al. 1980). Although these data are often cited as evidence of students' inability to estimate or of the complexity of estimation, low scores may also reflect limited experience and the long-term effects of a curriculum that presents a limited view of mathematics in which conventional procedures for manipulating symbols are emphasized. This curricular effect undoubtedly influences the type of thinking most students display when presented with estimation tasks in school contexts and makes it difficult to get an accurate picture of their capabilities.

Of even greater importance, perhaps, is the generally negative view students come to hold toward estimation, as described here (Trafton 1986, p. 16):

> Most students are uncomfortable with estimation. They are not sure what it is or why they need to do it. They find estimating cumbersome and often require paper and pencil and a great deal of time to produce an estimate. When students are required to estimate, they frequently work out the exact answer first and then round it to get an estimate. In fact, many equate estimation with conventional rounding. Further, and more fundamentally, they do not view estimation as legitimate mathematics; in their minds, mathematics deals only with exact answers. They often ask why they can't just find the "real" answer.

Traditionally, estimation has received cursory treatment in curriculum materials. Estimation instruction usually follows paper-and-pencil computation and receives little ongoing emphasis. It is often presented as computation with numbers rounded to multiples of powers of 10, and instruction emphasizes a single correct estimate. Teachers' views about estimation are similar to those of students. Little research has been done on estimation. However, the few studies that examine the effects of estimation instruction show that students' ability to produce estimates increases and that they learn a greater range of strategies (Schoen et al. 1981; Reys et al. 1984). The use of multiple strategies by good estimators is consistent with calls for the development of a broad range of estimation strategies with students (Trafton 1978; Reys 1986).

Some research has documented the reasoning of good estimators. One study (Reys et al. 1980) identified three key processes used by the subjects—reformulation, translation, and compensation—as well as other common characteristics they possessed. The study has been replicated in Japan with similar findings (Reys et al. 1991).

Because estimation research is limited by curricular effects and students' lack of experience, theories concerning the development of estimation (Case 1985) should be viewed with caution. Although there is a paucity of research directly on estimation to guide research and development, there is growing evidence from related areas of investigation that support developing children's knowledge in these areas. These data include work on children's natural thinking processes and intuitive mathematical understandings, findings from constructivist psychology, and work on situated cognition. These areas are addressed by Silver (chapter 13) and Sowder (chapter 12) in this volume.

Today, there is again strong professional support for a major role for estimation in the curriculum. One reason frequently cited is the recognition of estimation as a basic skill in a technological era in which individuals need to be able to judge the reasonableness of answers obtained with calculators and computers. An even more fundamental reason is found in the current mathematics education reform movement throughout the world in which an overriding goal is to develop mathematically literate citizens who possess mathematical power (National Council of Teachers of Mathematics 1989). Central to the idea of mathematical power is the ability to think and reason mathematically and to use mathematics to solve nonroutine problems. Mathematical power requires individuals to have a deep understanding of mathematical concepts and relationships, ones for whom learning mathematics is a sense-making experience.

Estimation, with its close links to number sense and problem solving, plays an important role in helping students make sense of mathematics and in attaining the goals of a revised curriculum consistent with the vision presented by the NCTM *Curriculum and Evaluation Standards for School Mathematics*. Estimation also finds support in the current call for a broad curriculum in which content is set in the context of applied situations.

Thus, the opportunity exists today to make substantial progress toward a strong emphasis on estimation in the curriculum and to realize the potential that has long been attributed to learning to estimate. It is useful to have clear direction from research on which to develop the curriculum. However, both research and development may need to proceed simultaneously in order for knowledge from good practice and research to inform and give focus to each other.

This paper examines three areas that can guide future research. The discussion draws upon knowledge about estimation and related areas and explores questions and promising directions.

GOALS FOR COMPUTATIONAL ESTIMATION

Research and curricular goals for computational estimation must be guided by a vision that describes what it means to know and use estimation. These goals will reflect our conception of estimation, will guide the formulation of research questions and design, and will influence curriculum development, instruction, and assessment. The task of setting goals is made more complex by the fact that estimation is not so much a specific body of knowledge as it is a multifaceted process with many of the same subtleties and complexities as problem solving and number sense.

One broad set of goals is presented below and organized in three clusters. These clusters are not disjoint, nor is the list exhaustive. As a result of their experiences with computational estimation, we want students to be able to do the following:

Goals related to knowing and doing mathematics

- View estimation as legitimate and useful mathematics.
- Make informed quantitative judgments about computational situations.
- Demonstrate confidence in their ability to estimate.

Goals related to producing an estimate

- Use a variety of techniques.
- Use strategies appropriate to various situations.
- Justify estimates and solution processes.
- Recognize the relationship of an estimate to the exact solution (overestimates and underestimates).
- Make judgments about the precision required in estimates.
- Modify initial estimates to give more precise estimates when appropriate.

Goals related to using estimation in exact computation

- Recognize whether exact solutions are reasonable or not.
- Use estimation to monitor the execution of exact computation.
- Make judgments about the precision of estimates required.

These goals are consistent with the overall goals stated in the *Curriculum and Evaluation Standards* for all students: (1) that students become mathematical problem solvers, (2) that they learn to communicate mathematically, (3) that they learn to reason mathematically, (4) that they learn to value mathematics, and (5) that they become confident in their ability to do mathematics (NCTM 1989, p. 5).

These goals for estimation also more adequately portray the nature and characteristics of computational estimation than more traditional conceptions of it do. They move it beyond the acquisition of a specific set of narrow behaviors, such as knowing or using specific techniques to produce estimates. Further, estimation is viewed in a manner similar to problem solving, with which it shares many characteristics.

Finally, this set of goals can help define and focus future research and evaluation efforts. For example, most evidence of estimation ability is based on producing or selecting an estimate. Although such evidence is one element in evaluating student understanding and performance regarding estimation, it is not sufficient when viewing estimation in the manner discussed above.

INSTRUCTIONAL CONTEXT FOR COMPUTATIONAL ESTIMATION

A second major consideration is estimation's instructional context. Historically, estimation has been an "add on" to instruction in standard computational procedures and a preliminary to exact computation. Usually, estimation tasks involve abstract computation. Thus, no clear purpose for estimating is established. These instructional characteristics may account for the way students view estimation. Development of estimation should address three elements: the computational process, applied situations, and emphasis on number sense.

The Computational Process

Figure 7.1, an adaptation of the *Standards* diagram (see also fig. 1.1), presents a comprehensive view of computation and highlights estimation's role in computation (Rathmell and Trafton 1990). Because estimation is presented as one of two computational alternatives for problems, it receives direct attention as a legitimate goal in itself. This matches the use of estimation in daily life and other applied situations.

For example, shoppers estimate when comparing prices of two items or when determining whether one has enough money to purchase three items. Exact computation is performed by a cash register or not at all.

The dotted line from "estimate" to "exact answer" (see fig. 7.1) shows a relationship between estimation and the process of carrying out exact computation. It may be useful to view estimation as serving a monitoring function that can occur at three places: (1) *Before* exact computation, estimation can establish a ballpark sense of the answer. (2) *During* computation, estimation can monitor whether the work is moving in the right direction. (3) *After* computation, estimation helps one sense whether the answer is sensible or reasonable. Precise estimates need not be made at any of these three stages. For example, common sense may tell us that a result falls outside of a possible range: The total of items costing $1.29, $1.29, and $0.98 "just can't be" as much as $4.68.

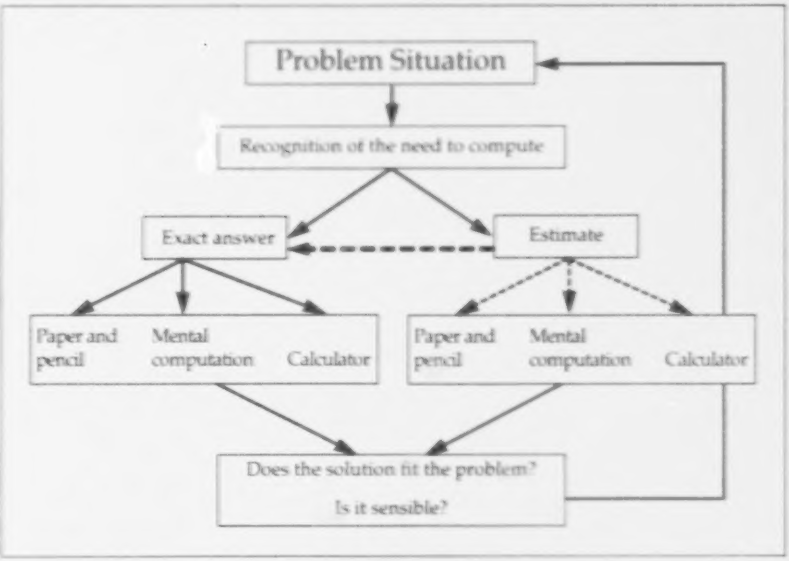

Problem Situation
Recognition of the need to compute
Exact answer
Estimate
Paper and pencil
Mental computation
Calculator
Paper and pencil
Mental computation
Calculator
Does the solution fit the problem?
Is it sensible?

Fig. 7.1. Computational model

Hiebert (1984) argues that estimating "encourages one to step back, to think about the structure of the problem, and to focus on the reasonableness of the solution" (p. 510). Despite the often cited importance of this role for estimation, there are few, if any, studies that have examined it in detail. It would be helpful to know how insightful estimators actually use estimation when carrying out exact computation, how often they do so, and how students would estimate if they were consistently trained in estimation.

Two studies suggest that students have difficulty with using estimation to judge the reasonableness of computations. Reys et al. (1980) asked students to perform computations on a calculator that was programmed to give successively larger errors. The results showed that students generally were reluctant to challenge the calculator result. Vance (1986) conducted a study dealing with the multiplication of decimals in which estimation was part of the instructional program. Nonetheless, when presented with the task of locating the decimal point in the product for 358.8 × 0.25 and given the digits of the product minus the final 0 (8965), 80 percent of the students chose 8.965. They apparently used the conventional rule for counting decimal places in the product and ignored the fact that the product was not reasonable. However, classroom teachers who emphasize estimation often report that students become better at recognizing unreasonable solutions.

Applied Situations

Figure 7.1 also emphasizes that computation should be used to solve problems. The related notion of linking mathematics to the world of the students is a predominant aspect of elementary school mathematics reform (NCTM 1989, p. 16). There is abundant evidence that children and adults in familiar situations think flexibly, inventively, and at a high level when performing number and computational tasks (Lave, Murtaugh, and de la Rocha 1984; Resnick 1988).

Here is an example described in the K–4 Standards (NCTM 1989, p. 37): Three children were discussing the purchase of clothing shown in an advertisement. Two children selected items, and the third child entered the prices into a calculator. After examining the calculator result, one child noted that the total cost could not be more than the $50.00 shown on the calculator because the two shirts were less than $14.00 each and the pants were $17.99. The context of applied situations also influences the reasoning employed. When adults and children are asked to find the cost of nine notebooks that cost 25 cents each, they often make use of the fact that four 25s is $1.00. That thinking is seldom employed when asked to find the product for 9 × 25.

In summary, it seems that students learn estimation best when consistently faced with practical problems that would normally prompt them to estimate. Schoen and Silver provide additional support for this position in chapters 8 and 13.

Emphasis on Number Sense

The importance of carefully developing an understanding of number concepts and relationships has long been recognized. One principal aspect is emphasizing the development of number sense. The NCTM *Curriculum and Evaluation Standards* discusses number sense in the context of developing intuitions about number relationships and includes in its description of number sense such components as well-understood number meanings, the development of multiple relationships among numbers, and awareness of the relative magnitudes of numbers.

Howden (1989) suggests that number sense "develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (p. 11). She cites examples of the responses she received when she asked first-grade children to tell what came to their minds when she said "twenty-four." These include "two dozen eggs," "four nickels and four pennies," "take a penny away from a quarter," "take six pennies from three dimes," "I'll be twenty-four in nineteen years," and "Twenty-four is when the hand (on a scale) is almost in the middle of twenty and thirty." The children made use of various partitions or groupings of twenty-four in their responses and showed an understanding of various relationships (if 25 minus 1 is 24, then 30 minus 6 is also 24). Intuitive understandings about doubling and halving numbers permit them to suggest that twenty-four is also a little less than half of fifty.

When models are used with fractions, children describe similar relationships. Two-fifths, for example, is a little less than 1/2, a little more than 1/3, almost the same as 3/8 (maybe a "tad" more), and exactly the same as 4/10. Further explorations can lead to the understanding that 3/4 is halfway between 1/2 and 1, is the same as a half plus a fourth, is 1/4 less than 1, is 1/8 less than 7/8, and so on.

Responses involving partitions and number relationships occur regularly in children's self-generated mental computation strategies. Second graders often use one of the following ways of partitioning 250 when asked how to share 250 items with another child: "100 and 100 is 200, and that leaves 50. 25 and 25 is 50." "Half of 200 is 100. Half of 50 is 25; so 125." Third graders, when asked how much change is left from $1 after a 49-cent purchase, frequently use the fact that 50 and 50 is 100 to solve the problem.

"Good" estimators across a range of ages deal with number in flexible ways (Reys et al. 1980), as shown in the example below (p. 197):

> *At the 1979 Superbowl, 8,483 hot dogs were sold for 60 cents each. About how much resulted from selling the hot dogs?*
>
> Subject A: Used 8,000 as opposed to 8,500 ("Thousands are easier to multiply") and multiplied by 60 cents—$4,800. "A little more than that."

Subject B: "Round the number up to 8,500 and the price to 50 cents each, so 8,500 × 0.5 gives $4,250. It should be a little more because of knocking off the dime—say it's $5,000."

Subject C: Split $0.60 into 0.5 and 0.1; took 1/2 of 8,000 and added 1/10 of 8,000 to give 4,000 + 800, or $4,800.

A common thread that runs through number sense, mental computation, and estimation strategies is an approach that involves the "manipulation of quantities" rather than the "manipulation of symbols" (Carraher, Carraher, and Schliemann 1987). There is a physical quality to the thinking. For example, children "break" 250 into a 200 part and a 50 part, operate on each of the parts, and then combine the results. Although this action can be explained in terms of the properties of the real number system, I do not believe this properly characterizes their thinking.

Thus, a strong argument can be made that certain thinking processes are common to number sense, self-constructed mental computation algorithms, and computational estimation. These strategies involve flexible ways of thinking about numbers and a reliance on intuitive number relationships. It is plausible that children will more likely be able to access and use such strategies if they have consistently been a part of their experiences with number and computation.

In summary, estimation needs to be part of a total approach to developing number and computation. Attaining goals for estimation is not the only result. Successful teaching involving estimation also uses activities that promote thinking strategies and create a classroom environment that encourages the development of number sense. These considerations suggest caution in attempting to analyze and discuss computational estimation apart from these factors. I believe that estimation is increasingly difficult to separate from a collection of variables of which it is an integral part.

A MODEL FOR CURRICULUM DEVELOPMENT AND INSTRUCTION

Building estimation into the curriculum and providing productive instruction is a major task for several reasons: (1) estimation has had a limited role in the past, (2) estimation and its significance are not well understood, (3) traditional views on computation instruction may retard instructional efforts to teach estimation, and (4) teachers have limited classroom experience with which to guide development. However, a body of research exists on which development efforts can draw. Additional evidence from classroom experience can also help to sharpen the focus of future research. This situation is similar to that of problem solving and number sense, namely, there is a lack of agreement in these areas about their nature and an initial lack of clear direction. However, this lack of direction and agreement on the content of these areas should not keep us from moving ahead. The following discussion proposes a model for helping students develop computational estimation.

Build a Solid Base

We must provide experiences that develop the kind of understandings and mind-set that facilitate estimation. One skill necessary for good estimation is number sense, for example, describing and partitioning numbers in a variety of ways. One exercise compares the relative sizes of numbers: 31 is … large compared to 4, small compared to 89, about the same as 28, and about half of 60. Such activities build a deep understanding of number and flexible ways of thinking about numbers.

Estimating quantities, another component of number sense, is also useful. For example, children recognize that a 17-link chain is between reference chains of 10 and 20 links, that it is closer to 20 links, and that it is 3 links less than 20. Similarly, children can use a 15-link reference chain to roughly estimate the length of a 35-link chain. Such experiences also build a sense of the pervasiveness and usefulness of estimation. Children's experiences with creating and sharing mental computation strategies are also important in building a solid base for computational estimation.

Provide Early and Varied Experiences

Figure 7.2 shows a model for the learning of many topics. The model suggests that children build formal mathematical knowledge on their natural encounters with mathematical ideas in a variety of contexts and have extensive informal experiences with a mathematical idea. The first box emphasizes the importance of informal experiences with number and quantitative estimation described above. These experiences would include the use of estimation language such as *about*, *a little more than*, *between*, and *close to* for describing quantitative experiences. The second stage includes estimation activities involving informal or reasonable quantitative judgments. For example, there are two boxes of marbles, one containing 46 marbles and the other containing 37 marbles. Students might be asked whether they think there are more than 50 marbles altogether, or less than 100 marbles, or how they know there are more than 70 marbles.

This type of activity, which often makes use of reference points to give focus to the thinking, is related at a later stage to producing an estimate. At this level it is more of a "let's think out loud about this situation" type

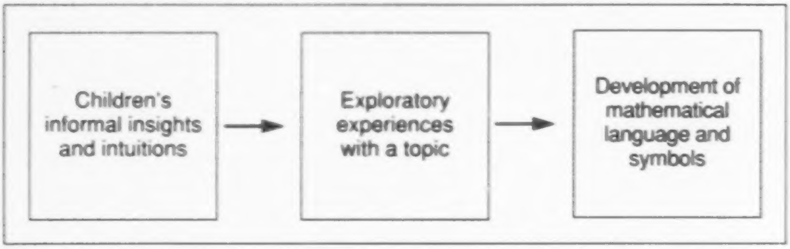

Children's informal insights and intuitions
Exploratory experiences with a topic
Development of mathematical language and symbols

Fig. 7.2. Model of the building of formal knowledge based on children's natural encounters with mathematical ideas

of activity. Similar examples might involve such situations as determining which two items a person could purchase with a given amount of money, or determining whether a person who has read 48 pages of a 100-page book has read more or less than one-half of the book. Many parallel situations can be used with percent. Many of the popular "target" calculator activities can also be viewed as "quasi-estimation" experiences that involve conversations about what represents a reasonable quantitative judgment in a situation.

If children have a variety of experiences—both planned and extemporaneous—in early stages of work with number, they will likely become familiar with the type of thinking used in making an estimate. By seeing estimation as a particular type of "conversation about number," they learn the many dimensions of the broader construct of estimation described in the section on goals. Again, these are ideas that can become the focus of systematic research investigations.

Develop Computational Estimation

The previous two sections provide essential readiness and are an integral part of the development of computational estimation. This section focuses more directly on the development of strategies for producing an estimate and several factors associated with this process.

First, research suggests that successful estimation requires a curriculum that is fundamentally different from the current one. "Patching" estimation onto our curriculum—with traditional emphases and traditional treatment of number, operations, and computation—is not likely to be successful. Because estimation is a way of thinking about quantitative relationships, children's prior experience must encourage and support this type of thinking so that children can access and develop it. Research on children's beliefs about mathematics would tend to support this position.

Second, although there are times when it is necessary to focus directly on estimation, it also seems important for estimation to arise naturally and to be addressed informally in an ongoing fashion. This way, it is more likely that children will see it as a natural aspect of computation and integrate it more fully with their other computation experiences.

Third, estimation, together with mental computation, can be part of early computation work and can be developed before paper-and-pencil computation. For example, work with percent is a most difficult area of the curriculum. However, developing an understanding of percent can build on children's intuitions about 0%, 50%, and 100%—including their fraction connections (Allinger and Payne 1986). Informal testimony of teachers who have experimented with this approach suggests that it could be a fruitful area for further investigation. Not only does an initial emphasis on mental computation and estimation have the potential for helping students see the power of these tools, but it also has the potential for bringing added meaning to conventional procedures.

Finally, there is the question of how children acquire strategies for producing estimates. Traditional views, and some early development efforts, identify a strategy, teach it to students, and provide practice on it. From a constructivist's perspective, it is more productive to provide estimation situations and encourage children to share the thinking strategies they employed. This method is similar to that being advocated for mental computation. The bias of the author on this matter is reflected in his choice of the word *develop* rather than *teach* in the title of this section.

Classroom experiences with this approach to mental computation suggest that it is an exciting and powerful way to build competence with producing estimates and to develop insight into the total process. This approach also addresses the overall goals for school mathematics programs today. Moreover, this method also makes it possible to give attention to particularly useful strategies as they arise.

Although much remains to be accomplished, there are reasons to be optimistic about making strong progress toward the kinds of programs that are needed rather than pessimistic about futile past efforts and the poor performance of today's students.

8

Assessing Computational Estimation: Research and New Directions

Harold L. Schoen

COMPUTATIONAL estimation is an important goal of mathematics instruction made even more important by current advances in technology and constructivist views of learning. However, assessing computational estimation is problematic. Although some research and development has been done on assessing estimation, most efforts have been made through traditional paper-and-pencil testing. Some aspects of estimation can be tested in traditional ways, but each of these methods has unique strengths and limitations. Research helps us understand these boundaries, but it has not provided a way to assess computational estimation that most mathematics educators would consider to be valid.

Today's curriculum reform is challenging researchers and test developers to view estimation as one technique in a rich and flexible set of mathematical problem-solving tools. This is a challenge and, as this paper will argue, one for which traditional measurement techniques are clearly inadequate.

In the "What Has Happened?" section of this paper, several attempts to test estimation in large-scale testing projects are described. In the "What Is Happening" section, the last decade or so of developments in the United States are reviewed. These studies describe estimation processes used by good estimators. They also describe the strengths and limitations of various ways to test whether students can use these processes. In "Future Directions," a rationale is put forth for a broader, context-bound view of estimation and its performance-based assessment.

WHAT HAS HAPPENED?

In the 1960s, an early attempt at testing computational estimation was made in the National Longitudinal Study of Mathematical Abilities (NLSMA). A sample NLSMA estimation item for sixth graders is given in figure 8.1. However, NLSMA researchers warned that such items may not be testing estimation, and they found evidence that many students performed exact computations (Wilson, Cahen, and Begle 1968).

If there are 5,280 feet in one mile, how many feet are there in 7 miles?

A. 35
B. 350
C. 3,500
D. 35,500
E. 350,000

Fig. 8.1. A sixth-grade estimation item from NLSMA

Since NLSMA in the 1960s, few national testing programs have included estimation in a significant way. For example, Reys (1986) analyzed the estimation content in the computational subtests of five of the most widely used nationally standardized tests: California Achievement Test, Iowa Test of Basic Skills, Metropolitan Achievement Test, SRA Achievement Series, and Stanford Test of Academic Skills. At most, only two or three of the forty to fifty items on each test were meant to involve estimation, and Reys questioned whether estimation was really required for some of those items.

Beginning in 1972, the National Assessment of Educational Progress (NAEP) has included computational estimation items in each of its four administrations to national samples of children of several ages. With regard to these items, however,

> there is no way to tell what strategies the students used. Rather than arriving at an estimation on the basis of mental computation using some rounding or 'sensing' the relative sizes of the numbers in an item, students may have used the less efficient means of doing the exact paper-and-pencil calculations and then choosing the number that was closest to their answer. (Kouba, Carpenter, and Swafford 1989, pp. 88–89)

Table 8.1 shows a sample NAEP item with a summary of student responses. Concerning students' performance on NAEP estimation items, Kouba, Carpenter, and Swafford (1989) reached the following conclusions: (1) students' errors on most whole number estimation application items appeared to be more a result of misinterpreting the items than of making errors in estimation; (2) there was evidence that some students lacked an understanding of the relative size of numbers greater than 100; and (3) in general, students did poorly on the items that involved estimating using factors, percents, or square roots. In short, test developers are dissatisfied with the validity of their estimation measures, and, to the extent that estimation is measured, student performance is weak.

There are many barriers to validly testing whether students can and do make appropriate use of estimation in computation situations. First, test

Table 8.1
Whole Number Computation Estimation from NAEP

	Percent Responding[a]	
	Grade 7	Grade 11
What is the estimated total weight of 185 lbs., 107 lbs., 216 lbs., 204 lbs., 169 lbs., and 168 lbs.?		
800	7	3
1000*	54	78
1300	18	12
2000	10	3
12,000	8	2
I don't know.	4	2

[a] Response rates were .80 for grade 7 and .98 for grade 11
* Indicates correct answer

writers must insure that students estimate rather than perform exact computation. A common, if inadequate, solution is to allow students only a few seconds for each item in the hope that they will not have time for exact computation. This can be done by administering the test on a computer, overhead projector, or slide projector. Such procedures appear to eliminate most exact computation but also cause students to make decisions and apply their estimation processes quickly and perhaps less meaningfully than they might with more time. Furthermore, estimation in problem-solving situations cannot be measured this way, since students need time to comprehend the problem.

A second validity issue arises when estimation is viewed as a flexible set of processes. In this view, a valid test of estimation should measure whether students know and use more than one estimation process. Traditional tests do not provide process data.

Scoring is a third issue. Because there is no single correct answer to an estimation problem, more than one appropriate estimation process may lead to more than one "correct" estimate, which makes scoring difficult. In a multiple-choice format, the challenge is to write the foils to credit students using a variety of appropriate estimation strategies.

These issues have been addressed recently by researchers, and some of the results are discussed in the next section.

WHAT IS HAPPENING?

This section has four subsections, each dealing with an area of consensus among researchers studying the assessment of computational estimation: (1) Estimation ability is inextricably linked to the understanding of number and operation concepts. (2) With no prior estimation instruction, many students are puzzled by test items that tell them to estimate. Most students who do try to estimate rigidly view computational estimation as limited to rounding the given numbers to the closest leading power of 10 and then mentally computing with the rounded numbers. (3) Different estimation processes can be tested in isolation by manipulating the item formats and, in multiple-choice items, the foils. However, issues of validity have not yet been satisfactorily resolved by any researcher or test developer. Performance assessment may be the most valid approach, but little has been done in this area. (4) There is growing evidence to suggest that placing estimation in a context, especially one that is familiar to the student, has a strong, positive effect on student performance (see Silver in chap. 13).

Estimating and Understanding Numbers

For purposes of instruction and research, computational estimation suffers from the lack of a clear definition. It is related to, but different from, paper-and-pencil computation, mental computation, and number comparison. It occurs, or should occur, in a problem-solving context. The process of estimation itself is problem solving. Good "number sense" implies good estimation skills. Except perhaps for paper-and-pencil computation, all these estimation-related topics require a broad understanding of numbers, including their relative magnitudes, their structural properties, their use in operations, the effects of those operations on magnitudes, and the use of numbers and operations to represent quantities and transformations in the real world.

Descriptions of processes used by good estimators partially illustrate the connection between estimation and understanding number and operation concepts. Besides rounding by the usual textbook rules, good estimators also make appropriate use of front-end estimation, compatible numbers, averaging or clustering, and adjusting or compensating for other rounding (Reys et al. 1982; Reys 1984). Good estimators examine the numbers and other conditions in the setting and decide on an estimation strategy. For example, if only a rough estimate is needed, it may be sufficient to compute mentally with the leading digits. If a sum of *n* roughly equal numbers is required, a good estimator may decide to multiply mentally the "average" number times *n*. A good estimator who rounded two numbers downward to estimate a product may adjust the estimate upward a bit to compensate. In each case, the good estimator clearly illustrates an understanding of number and operation concepts, but there is still a "one to many" relationship between an estimation problem and its appropriate processes and outcomes.

In explaining the role of mental computation and number comparison in developing number sense, Sowder (1988) notes that

> the relationship among mental computation, number comparison, and computational estimation can be found in their common heritage of dependence on place value concepts, on the ability to work with powers of 10 and to use properties of number operations, and on the understanding of the symbol system used to represent numbers. (P. 195)

The nature of the relationship Sowder identifies is still not entirely clear, although mathematics educators interested in estimation agree that it exists (e.g., Trafton 1986; Carlow 1986; Behr, Post, and Wachsmuth 1986). That an understanding of number and operation concepts is a corequisite of good estimating cannot be ignored in assessing estimation. A valid test of computational estimation must also measure some aspects of understanding number. Recognizing this, researchers in estimation assessment describe their test as Estimation and Number Concepts (Schoen, Blume, and Hart 1987; Schoen, Blume, and Hoover 1990). Also, developers of the Iowa Test of Basic Skills plan to include a significant number of estimation items in their Concepts, rather than Computation, subtest. Unfortunately, students do not generally recognize the connection between estimation and understanding number and operation concepts, a fact discussed in the next section.

Students' Beliefs about Estimation

The confusion many students experience when faced with an estimation task is well documented (e.g., Kouba, Carpenter, and Swafford 1989; Schoen, Blume, and Hart 1987; Schoen, Blume, and Hoover 1990; Sowder 1988; Wilson, Cohen, and Begle 1968). This confusion appears to stem from two main sources.

First, many students have had little or no instruction in estimation and are stymied by the tasks' directions. During interviews, students report that they have never done estimation or that they don't know the meaning of words like *estimate* or the *answer is about....* The confusion sometimes persists even with previous instruction in estimation.

The second source of confusion, which may help explain the first, appears to be conflict between the estimation task and the student's strongly held belief that there is only one algorithmic way to solve every problem, which results in a single right answer. Learning experiences in estimation are necessary to change this view. On a small scale at least, instruction in estimation has been effective (e.g., Benton 1986; Case and Sowder 1990). (However, none of these researchers claim to have changed the students' belief that mathematics problems have only one right answer. Rather, they have been able to greatly reduce students' confusion when presented with an estimation task.) Good estimation instruction, however, is not widespread in U.S. schools. So, large-scale efforts to test estimation, independent of local classroom programs in estimation, will be limited by students' unfamiliarity with estimation tasks. Because students believe in one right algorithmic method and one right answer, they often prefer the standard, algorithmic rounding approach: (1) using the standard rules for rounding, round the numbers to be computed to the nearest leading power of 10, (2) mentally compute with the rounded numbers, and (3) call the result an estimate (Schoen, Blume, and Hart 1987; Sowder 1988). Because this approach is the one most commonly included in textbooks, students' classroom experiences with estimation probably reinforce their preference for the rounding algorithm. In fact, Schoen et al. (1981) found that most students in grades 3–6 still reverted to rounding on estimation tasks after two weeks of instruction in other estimation strategies. The researchers also found that very few students could make a second estimate that they thought would be closer than their first. "They had already used the only estimation process they thought was 'right' to get the first estimate" (Schoen et al. 1981, p. 177). The idea of compensating for an estimate was foreign to these students.

The strength of the "rounding" preference and the importance of the issue for large-scale test development are illustrated by the item in figure 8.2 (Schoen, Blume, and Hart 1987).

	Percent Responding by Grade			
Grade	5	6	7	8
The closest estimate of 98 × 2.62 is ___.				
a) 90 × 2.5	8	6	6	3
b) 100 × 2	13	14	9	7
c) 100 × 3	42	68	71	74
*d) 100 × 2.5	28	12	11	14

Fig. 8.2

The correct answer, (d), was chosen by only 16 percent of 1376 students in grades 5–8. The most popular choice was (c). When presented with this item in interviews, students made similar choices, which they arrived at by rounding to 100 × 3. One student even noted, "You can't round 2.62 to 2.5." As the grade level increased, fewer students chose the correct answer, and more students chose (c). Furthermore, the biserial correlation of performance on this item with the entire 30-item test was −.41 in grade 8 compared to .26 in grade 5. Schoen and his colleagues found the same trend across similar items. These results suggest that older students who did well overall on the test are particularly consistent in their use of rounding in estimation tasks.

For traditional test developers, performance on this item is also distressing because it has a strong negative effect on the reliability of the test. However, research has much to offer test developers concerning the strengths and weaknesses of estimation items in various formats. This information may help improve the reliability and validity of tests.

Estimation Test Item Formats

Several testing approaches and item formats have been used to test computational estimation. Each has limitations. In one approach, items are open-ended, with limited response time allowed on an item-by-item basis so that students cannot compute exact answers. Reliable tests have been written and administered in this way (Reys et al. 1982; Schoen et al. 1981; Schoen, Blume, and Hart 1987). On the negative side, most mathematics educators would not want to limit the meaning of estimation to such tasks, since they probably fail to allow students time to apply much understanding of number concepts and are useless for estimation in problem solving. Furthermore, Schoen, Blume, and Hart (1987) found that about 70 percent of students apply the standard rounding algorithm and that other valid estimation processes are rarely used. Thus, these tests really only test one estimation process.

Several multiple-choice formats have also been used to test estimation, and reliable tests have been constructed from such items (Schoen, Blume, and Hart 1987). Research suggests, however, that different multiple-choice formats require estimation processes and yield student performance data that vary in ways that may be important for both test developers and teachers. One such study, Schoen, Blume, and Hoover (1990), compared student performance on carefully matched estimation items in five multiple-choice formats. The study included large-scale testing (1376 Iowa students) and individual interviewing of students in grades 5–8. Figure 8.3 gives an example of an item in each format: standard multiple-choice (ST), operation in foil (OF), range in foil (RF), benchmark (BM), and order of magnitude (OM).

The results showed that the formats, in order from easiest to most difficult, were OM, OF, ST equal to BM, and RF. Estimation processes used by students also varied by number type and format. By carefully constructing the foils, the researchers were able to elicit the following processes from students: (1) standard rounding, (2) front-end rounding, (3) other rounding, such as combinations of 1 and 2, (4) compatible numbers, and (5) compensation. Schoen and his colleagues wrote standard multiple-choice items that elicited all five of these processes. Range in foil items elicited all but the fourth process. Of course, not every standard multiple-choice item elicited all five processes. Only two estimation

(ST) The closest estimate of
1,926 + 851 + 3,272 is

a) 5,000
*b) 6,000
c) 7,000
d) 13,000

(OF) The closest estimate of
1,926 + 851 + 3,272 is

a) 1,000 + 1,000 + 3,000
*b) 2,000 + 1,000 + 3,000
c) 2,000 + 1,000 + 3,000
d) 2,000 + 8,000 + 3,000

(RF) 1,926 + 851 + 3,273 is
between

a) 4,000 and 5,500
*b) 5,500 and 6,500
c) 6,500 and 7,500
d) 12,500 and 13,500

(OM) The closest estimate of
1,926 + 851 + 3,273 is

a) 600
*b) 6,000
c) 60,000
d) 600,000

(BM) Is 1,926 + 851 + 3,273 more or less than 7,000?

a) Less, because 1,926 + 851 + 3,273 is less than 2,000 + 8,000 + 3,000
*b) Less, because 1,926 + 851 + 3,273 is less than 2,000 + 1,000 + 4,000
c) More, because 1,926 + 851 + 3,273 is more than 1,000 + 8,000 + 3,000
d) More, because 1,926 + 851 + 3,273 is more than 1,800 + 800 + 3,000

Fig. 8.3. Matched items in five multiple-choice formats

processes were used on benchmark items (processes 1 and 5). Two others were used on order-of-magnitude items (processes 1 and 4). Students initially used standard rounding on items in the operation-in-foil format and then chose the foil that was most like the result.

Schoen and his colleagues concluded that all five of the formats test some aspects of estimation and can be used to construct reliable tests. However, test developers should be aware of outcome and process differences due to item format. Also, benchmark items, as used here, should only be used with the understanding that their relative difficulty is due in part to errors in logic and not only to students' inadequate estimation skills.

Most of the estimation items we have considered so far are purely symbolic. Testing estimation within a context is the subject of the next section.

Estimating Symbolically Versus in Context

Reys (1986) noted that student performance on an estimation item can change dramatically depending on whether a context is provided. This author verified Reys's observation in a recent unpublished study, using a design similar to that used by Schoen, Blume, and Hoover (1990), except that items were matched according to whether they were in a context or purely symbolic, whereas the earlier study had matched items across five formats. A sample of 1226 Iowa students in grades 5–8 were administered one of two test forms so that each item of thirty pairs was completed by a random half of the sample. A sample matched pair of items is given in figure 8.4.

The closest estimate of
98 x 2.62 is

a) 180,000
b) 18,000
c) 1800
d) 180

98 Flashlights

$2.62 each
The closest estimate of the total cost
of the flashlights is

a) $180,000 c) $1800
b) $18,000 d) $180

Fig. 8.4. Matched pair of symbolic and contextual items

The context made the items easier for students at each grade level. The data in table 8.2 are based on about 150 students per cell. The differences between means on symbolic and contextual items in each grade are significant at the .05 level.

Table 8.2
Mean Percent Correct on 30 Symbolic versus 30 Matched Contextual Items by Grade Level

	Grade Level			
Item Type	5	6	7	8
Symbolic	42.5	48.4	54.1	61.4
Context	48.1	55.7	58.6	64.3

Contexts that were familiar to students, especially those involving money, contributed most to the differences. No attempt was made in this study to find reasons why the contextual items were easier. It seems likely, however, that contexts for some items provide students with a concrete framework in which they can visualize the estimation, whereas symbolic items provide no such framework. Context-free processes like compensation and compatible numbers focus on the numbers or numerals only. When students estimate in a context, perhaps they supplement, or sometimes even supplant, these processes with context-specific estimation processes that make use of mental images of concrete objects or of concrete transformations. The research is certainly not conclusive, but such a point of view is consistent with two current trends: (1) that number sense (which includes estimation) is highly context-specific, and (2) that assessment of estimation skills among other mathematical concepts and skills should take place in a problem context. These trends are discussed further in the next section.

FUTURE DIRECTIONS

As a curriculum goal, computational estimation has received renewed support in recent national recommendations (e.g., National Council of Teachers of Mathematics [NCTM] 1989; National Research Council 1989). Because of advances in technology and changes in our view of what mathematics is and how it is learned, estimation is an important mathematical tool that all students need. According to the NCTM *Curriculum and Evaluation Standards,* the greatest revisions needed in teaching computation include the following (NCTM 1989, p. 95):

- Fostering a solid understanding of, and proficiency with, simple calculations
- Abandoning the teaching of tedious calculations using paper-and-pencil algorithms in favor of exploring more mathematics
- Fostering the use of a wide variety of computation and estimation techniques—ranging from quick mental calculation to those using computers—suited to different mathematical settings
- Developing the skills necessary to use appropriate technology and then translating computed results to the problem setting
- Providing students with ways to check the reasonableness of computations (number and algorithmic sense, estimation skills)

Paper-and-pencil computation is no longer the focus of elementary school mathematics. Estimation's place is as one part of an integrated view of computational proficiency. The central role played by estimation in this view can be seen in figure 1.2 in the introduction to this monograph by Reys and Nohda. Concerning how students learn mathematics, "Two general principles have guided our descriptions: First, activities should grow out of problem situations; and second, learning occurs through active as well as passive involvement with mathematics" (NCTM 1989, p. 9). Students are thought to learn mathematics through active involvement with mathematical ideas in a problem-solving context.

> Thus, instruction should vary and include opportunities for—
>
> - appropriate project work;
> - group and individual assignments;
> - discussion between teacher and students and among students;
> - practice on mathematical methods;
> - exposition by the teacher. (NCTM 1989, p. 10)

This integrated, contextual, activity-oriented view of mathematics and how it is learned calls into question our traditional ways of defining educational objectives and of assessing student achievement. Much of the research on computational estimation, especially its assessment, has been influenced by what Resnick (1989) calls decomposition and decontextualization assumptions. "The decomposition assumption refers to the notion that competence can be completely defined by a collection of independent elements of knowledge or skill.... The decontextualization assumption refers to the idea that competence exists independently of the performances that it enables; that there is some pure or abstract form of knowing that remains intact no matter what the conditions of its use; that knowledge is fully defined as something inside an individual's head, independent of the situation in which the individual acts" (p. 35). Rather than decomposing estimation into discrete, context-free processes to be taught and tested individually, the NCTM *Curriculum and Evaluation Standards* suggests that it should be seen as part of a broader construct called number sense.

It is nearly impossible to define number sense in traditional terms, that is, within the confines of the decomposition and decontextualization assumptions. Number sense usually includes most of the estimation processes that have been studied in the research cited earlier. However, the understanding of number and operation, flexibility and inventiveness, and the need for estimation to occur in context are also principal components (e.g., Resnick 1989).

Deciding how to test number sense or estimation is difficult when viewed in this way. Traditional short-answer or multiple-choice tests with single correct answers are clearly insufficient. Alternative assessment methods, involving the observation of students or groups of students as they perform in problem-solving situations, appear to hold promise. In such approaches, a student or a group of students is given a rich problem situation appropriate for estimation. Observers then judge the student's or group's performance holistically with estimation as one component of the evaluation. Such assessment techniques have recently received a great deal of attention, but research data are not plentiful. However, if current visions of mathematics curriculum reform are to become a reality, it is crucial that valid assessment methods for important curriculum goals like estimation be developed and widely implemented.

Part 3

Similarities and Differences: Calculator Use in Japan and the United States

NATIONAL reports in Japan and the United States support the use of calculators, and, in general, the mathematics education leadership supports their use for classroom use. There seems to be a lack of broad-based public support for calculator use in elementary schools in both countries, but the support for calculator use in learning mathematics is stronger in the United States than in Japan.

In the United States, the National Council of Teachers of Mathematics calls for extensive use of calculators at all grade levels. In Japan, the use of calculators has been recommended since 1966, and the current national curriculum encourages more calculator use in grades 5 and higher. The Japanese maintain that school mathematics is conceptual in the early grades, and because of this conceptual emphasis in grades K–4, the Japanese do not see the calculator playing a major role at that level.

The sharp differences in recommendations related to calculator usage are reflected in current school practice. In Japan, there is little use of calculators in elementary school mathematics programs. In contrast, in the United States some schools make extensive use of calculators in all grade levels, even though it is possible to find elementary schools where calculators are not used.

Clearly the calculator use in Japanese elementary schools (which would be rarely witnessed in mathematics classes) is greatly influenced by central government policy, which sets the curriculum and external exams. Calculators are not allowed on local or national mathematics examinations in Japan, but this issue is being studied. Tests are also an important part of the educational system in the United States, and allowing calculator use on standardized tests continues to present challenges. Currently national examinations are being revised to reflect calculator use, and most states have developed mathematics tests that allow calculator use.

In addition to testing, the very limited use of calculators in Japanese elementary schools results from several other factors. The main factor is the teacher's view of mathematics. Japanese teachers prefer to teach theory of mathematics rather than applications. This tendency increases with the school level, and many teachers view the calculator as only a computational tool for applications. Another consideration reflects the value placed on computational proficiency in Japan. Since Japanese students perform well on written computational skills as well as on higher-order thinking, there is a fear that the use of calculators will undermine this accomplishment. Some Japanese educators are unconvinced that calculator use will contribute greatly to mathematics learning.

There is a common belief among elementary and middle school teachers in both countries that "basic skills" should be mastered before children are allowed to use calculators. According to this view, children should learn their basic facts and computational algorithms first; then, perhaps, the calculator might be less harmful.

In the United States most mathematics educators view the calculator as useful in learning mathematics rather than just a substitute for paper-and-pencil procedures. These educators view the calculator as a powerful instructional tool rather than a device that might short-circuit mathematics learning and result in students being less mathematically proficient. In Japan, some mathematics educators do support the use of calculators in schools, but the issue has been less widely explored and studied.

In the United States, many mathematics textbooks feature calculator-based activities, although they are rarely an integral part of the mathematics units. In Japan calculators are used in the national guidebooks for teachers, but they are not widely shown in student textbooks or practice books.

Promoting ways of effectively using calculators to facilitate mathematics learning is needed in both countries. It is agreed that before widespread, effective use of calculators can occur in schools, teachers will need more opportunities to learn how to use calculators to enhance mathematics learning.

This summary reflects the national picture of calculator use in elementary school mathematics. We would be remiss in failing to acknowledge the role of the individual teacher. In Japan as well as in the United States, creative and reflective teachers are bringing innovations into their classrooms. Using stated policy as a guide, teachers exercise considerable personal freedom in creating learning environments in their classrooms. Thus in both countries, teachers can be found who make extensive use of calculators in meaningful ways.

In Japan, calculators will become widely used in schools when and if (1) teachers' view of mathematics is changed from a theoretical to an applications orientation, (2) calculators are allowed on examinations, and (3) educational materials that incorporate calculators are prepared. In the

United States, the use of calculators will continue to differ widely from one school to another. More extensive use of calculators will occur when all examinations reflect calculator availability and textbooks feature the calculator as an integral part of instruction. Of course, textbook publishers will produce such books when the demand warrants it. Demand is created when textbook selection committees insist on calculators being seen as an important tool in doing mathematics.

Thus we are likely to see in the United States continuing public debate over calculator use in schools with striking differences in use among schools and school systems. In Japan, calculator use in schools is changing slowly. Currently, there is no mandate for the use of calculators in either country, and there is no foreseeable date when calculators will become an integral part of learning in mathematics classrooms.

9

Availability, Use, and Assessment Related to the Calculator

Terrence G. Coburn

CALCULATORS have helped drive and direct changes in school mathematics programs from elementary school to beyond secondary school. These changes typically have not been dramatic, nor have they happened quickly. Nevertheless, the increasing availability of calculators during the last twenty-five years has slowly but steadily changed mathematics curriculum, instruction, and assessment. This paper explores this evolutionary process by examining the question, What has happened with respect to calculators?

WHAT HAS HAPPENED?

In the early 1960s, some mathematics educators explored the use of an electric desktop calculator at the secondary school level in remedial mathematics classes. Students in these classes usually lack computation skills and need to be motivated to study mathematics. The calculator's focus in these classes was on practical applications. The calculators were too expensive (around $600) to be made widely available. Also, the prevailing thought was that not all students should have access to calculators—only those who by eighth grade had not become proficient at arithmetic.

Milestones

1972 Texas Instruments entered the consumer market with the Data Math Calculator—an "expensive" ($150), four-function, hand-held calculator. (College Entrance Examination Board 1989).

1975 The National Advisory Committee on Mathematics Education (NACOME) report calls for calculator use for all students in school mathematics starting in grade 8 (Conference Board of the Mathematical Sciences 1975).

1977 First use of calculators on the second mathematics assessment of the National Assessment of Educational Progress (Carpenter et al. 1981).

1980 NCTM position that publishers, authors, and test writers integrate the use of the calculator into their materials at all grade levels.

1989 NCTM *Curriculum and Evaluation Standards* says that "appropriate calculators should be available to all students at all times" (NCTM 1989).

The decade of the 1970s saw the cost of calculators plummet, the capabilities and features of calculators increase, and the obstacles to calculator use in school mathematics become clearly established. Despite its early promise to revolutionize school mathematics, the hand-held calculator has not yet permeated the school mathematics program. Calculator use in schools steadily increased during the 1980s, but barriers against full implementation have been slow to come down.

Early Obstacles to Widespread Calculator Use

Restrictions to full classroom use and barriers to test use arose as calculators became more widely available in society. Obstacles to their full implementation in school seem to stem from two major concerns: (1) questions about the calculator's role vis-à-vis both the mathematics curriculum and methods of instruction, and (2) questions about equity—the availability of calculators to the masses and unfair advantages related to the different capabilities of various types of calculators. Other, more specific issues can be subsumed by the above two basic concerns:

Lack of curricular alignment

- Paper-and-pencil skills need to be mastered prior to routine calculator use.
- The calculator is okay to use for checking homework but not for assessing student learning.
- Calculators are not allowed on tests. Until they are allowed on tests, we shouldn't use them in the classroom.
- Teachers need to be trained to teach mathematics using a calculator.
- Instructional materials need to incorporate calculator use.
- The calculator is an inappropriate crutch. Students will become dependent on calculators and not learn basic skills.
- The support of administrators and parents is difficult to obtain and clearly relates to one's perception of a proper mathematics curriculum. It often requires helping these groups become knowledgeable of the potential role for calculators in school mathematics programs.

Equity

- Not all students or schools can afford calculators.
- Some calculators have more capability than others, and students who are able to afford them should not have an unfair advantage on tests.
- Some calculators have large memory and programming features, which could breach test security.

The mathematics curriculum in which the calculator sought acceptance was dominated by computational procedures at the elementary and middle school levels and by symbol manipulation skills at the secondary school level. There was a conflict between what the calculator delivered and what many believed to be the goals of the curriculum. Thus, although the calculator proliferated at home and on the job, it did not find extensive use in the school. Some parents and educators were of two minds about technology. They were positive about computers because computers helped to deliver drill and practice, and they also believed that students needed to become computer literate. They also thought, however, that calculators should be restricted because students might become dependent on them and not attain proficiency in basic skills. This concern about becoming dependent on a calculator and losing proficiency on basic skills was, and still is, prevalent at all grade levels.

The second concern, equity, seems to have two main aspects: (1) availability and (2) calculator capability. As a function of cost, availability should be less of an issue today because costs have come down to an affordable level. Availability as a function of use is a curriculum issue and is thus connected to the prior discussion of curricular alignment. However, since in some parts of the United States some schools report difficulty affording heat, lights, and decent textbooks, the cost of providing calculators will remain a barrier to full implementation. Nonetheless, research studies and surveys during the 1980s indicate that students have access to calculators more at home than at school (Educational Testing Service 1988).

That calculators have differing capabilities raises questions of fairness and equity on tests. Validity is also compromised because a student's ability regarding the specific learning being measured may be masked by the calculator's power or by the student's lack of familiarity with the calculator.

Classroom use is also affected. Some calculators have features that may help the student to develop additional understanding of some concepts—for example, constant addition, fraction notation, graphing screens, or symbol manipulation.

Clearly, the issue of calculator capability interacts with the issue of curricular alignment. As calculators become more like versatile hand-held computers with large graphing screens and powerful subroutines, they further affect the question of what stays in the mathematics program and what should be deleted.

Assessment

Almost from the first appearance of the calculator in the classroom, many teachers and parents separated the issues of classroom use from the issues of test use. Although some limited role was awarded the calculator in the classroom—checking paper-and-pencil computation—calculators were initially barred from tests. Parents were willing to buy calculators primarily for use at home but were concerned that students might be at a disadvantage on standardized tests that did not allow calculator use. Calculators were barred from placement tests, standardized achievement tests, statewide assessment tests, and mathematics competitions, such as MATHCOUNTS. A number of very significant issues underlay the reluctance to admit calculators to the testing room—too much inequity and too little research has been cited.

The issues involving calculator use in the classroom and on tests are heavily intertwined, forming a vicious circle: calculators will be allowed on tests when they are more routinely used in the classroom, and they will be integrated into the classroom when they are allowed on tests. It is difficult to decide which influences the other more.

The National Assessment of Educational Progress (NAEP) first studied the use of four-function, hand-held calculators on tests in 1977–78 and has continued to incorporate calculator use on subsequent mathematics assessments. The fact that relatively few students used calculators as a regular part of their school mathematics may have had an adverse effect on their use on the NAEP tests. A short training booklet was used as part of the NAEP testing procedure, but it may not have been adequate training to produce effective calculator use on all appropriate items. Not surprisingly, students seem to perform better using calculators on routine computation items. However, NAEP and a variety of other studies have indicated that the results are mixed when contrasting calculator users and non–calculator users on various item types at different grade levels. Students at higher grade levels seem to perform better with a calculator than students at lower grades. Perhaps the older students can better relate their mathematical understandings to the operation of a calculator. On test items that require higher-level thinking, the calculator does not seem to provide any special benefit to the user. Also, there is a category of test items for which the calculator seems to be "irrelevant" or "neutral." The supposition that a calculator would provide the user with "quicker" procedures on the tests did not bear out. Perhaps some students were not familiar enough with the calculator to make efficient keystrokes. Some may have used the calculator when it was not the most appropriate tool for the job. These conjectures are all muddied by the lack of data on the relationship between the level of calculator training and the level of performance on some mathematics test items. In some cases, the data are reported as if the respondents used a calculator when in fact they may not have.

Allowing the use of calculators on current traditional standardized tests in order to study the effect on performance is perhaps a necessary step in the process of obtaining calculator use in schools, but it does not seem to make a very compelling case for calculator use on tests. After all, we expect students to perform better on items that require those characteristics that are delivered by calculators and not necessarily to perform better on items for which calculator use is not as relevant. Results have not been conclusive in part because calculator literacy is a confounding factor. Appropriate calculator training is not just a question of which key to press or which sequence of keys will result in the needed computational procedure; there is also the issue of when to use a calculator and when not to. Certain strategies with which the non–calculator user may not be familiar, such as guess and test, also must be taken into account.

WHAT IS HAPPENING?

Recent national reports—the NCTM *Standards* and *Reshaping School Mathematics: A Philosophy and Framework for Curriculum*—calling for major curriculum changes seem to be well received by educators, test developers, and textbook publishers. All concerned seem to be willing to move forward on the use of calculators in school mathematics. The barriers appear to be slowly coming down. Many teachers, administrators, and textbook authors seem to believe that calculators should be used in the classroom and that they eventually will be included on tests. Test developers seem to be proceeding with the development of new tests as if they believe that calculators will be in common use in the classroom.

Test developers are among the major players in curriculum change. The 1990 National Council of Supervisors of Mathematics program at Salt Lake City saw several test companies give presentations in which they reported on test development efforts involving standardized tests that allow the use of calculators. These are cautious steps, mainly involving pilot studies. In some studies, the test company had merely furnished a set of norms for calculator users. In many of the studies, students were allowed to use calculators on traditional tests—not tests designed for calculator use.

Progress toward calculator use seems to be more evident at the secondary school level. The Mathematical Association of America (MAA) is recommending that calculators (and computers) become an integral part of mathematics instruction and that they be used in placement testing programs. In 1991, the College Board announced the availability of Mathematics Level IIc, a new achievement test based on regular use of a scientific calculator in high school college preparatory mathematics. The College Board has also recently decided to allow the optional use of calculators on the new PSAT/NMSQT and the new Scholastic Aptitude (now "Assessment") Test (SAT 1) (Greenes and Rigol 1992).

The survey of teachers used as part of the 1990 NAEP indicates that calculators were seldom used (Educational Testing Service 1991b, p. 38):

> Calculator use appears relatively infrequent in fourth-grade classrooms. According to their teachers, only 6 percent of the fourth graders were asked to use a calculator several times a week and 47 percent were never asked to do so. Sixty-two percent of the fourth graders reported never using a calculator for mathematics.
>
> At grade 8, teachers reported that 30 percent of the students used a calculator at least several times a week, while 22 percent never did. Thirty-nine percent of the eighth graders reported never using a calculator in mathematics class, as did 14 percent of the twelfth graders taking mathematics.

A recent nationwide calculator survey of seventy-two mathematics education leaders was conducted, and forty people (55%) responded (Coburn 1990). The majority of these leaders are active in promoting the use of calculators in the classroom and on tests. Some results from this survey follow.

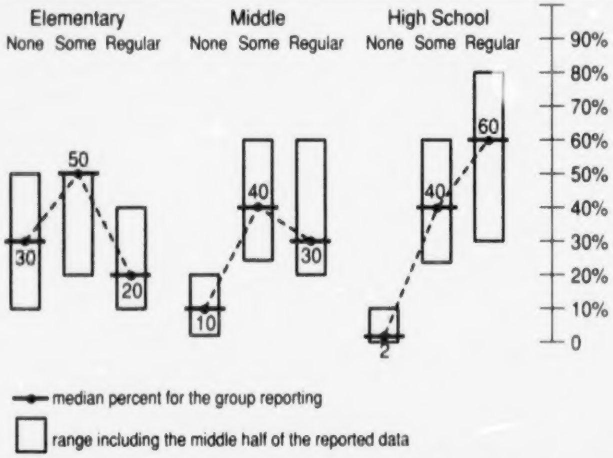
> "Do you see a trend in your geographic area toward increased use of calculators as part of school mathematics instruction?"

Ninety percent responded yes.

> "How much access to calculators do schools in your geographic area allow students?"

Respondents gave a percent estimate to indicate the proportion of schools allowing each of three arbitrary levels of access: None, Some, and Regular. The results are summarized in figure 9.1.

Participants in the survey were asked to rank some issues that could be obstacles to classroom implementation of calculators as a regular tool of mathematics instruction and for the evaluation of mathematics achievement. The results are summarized in table 9.1.



Elementary Schools. An average of 50% of the schools allow "some access." Only 20% allow "regular access."

Middle Schools. An average of 30% allow "regular access."

High Schools. An average of 60% allow "regular access."

Fig. 9.1. Percent of schools allowing access to calculators

Table 9.1
Issues That May Represent Obstacles to Classroom Implementation of Calculators (Mean average ranking)

	Elementary School	Middle School	High School
A. Too expensive	4.7	4.5	3.7 (62% ranked this issue 5 or 6)
B. Teacher training needed	2.3 (65% ranked this issue 1 or 2)	2.5	2.7
C. Lack of appropriate instruction materials	2.8	2.8	3.0
D. Calculators not allowed on standardized tests	2.7	2.7	2.8
E. Current curriculum does not call for calculator use	3.2	3.0	3.3
F. Perception that calculator use will be detrimental to student learning, to attaining basic skills, or to understanding	2.4 (60% ranked this issue 1 or 2)	2.8	3.8

Note: 1 = very significant; 6 = not very significant

Respondents were asked whether they felt that calculator use should be restricted in any way (for example, not allowed on tests) or whether the type of calculator used should be restricted in some way (for example, not using a scientific calculator in the primary grades). Various responses are summarized below:

Restrict calculator use:

a. No—23 percent
b. Leave to the teacher's judgment—13 percent
c. Restrict use on special tests (e.g., mental arithmetic)—30 percent
d. Two people said to teach the student how to make an appropriate choice.

Restrict calculator types:

a. No—40 percent
b. Leave to the teacher's judgment—20 percent
c. Four-function only in elementary school—20 percent
d. One person said that any calculator used should have AOS logic. Two people insisted on the constant addition feature.

This survey did not seek opinions regarding the grade level at which calculators should first be used; however, most data indicate that schools do not purchase calculators for students before grade 3. In part, this is an economic decision: Not having funds to supply all students with calculators, schools concentrate on the students who will use them most (those who do computation). This pattern also corresponds to the view that basic skills need to be learned first. In a recent survey of Missouri schools, 65 percent of the teachers surveyed felt that students should completely master the basic facts before they use calculators (Reys et al. 1990).

Statewide Mathematics Assessment Tests

Permitting calculator use on statewide assessment tests has legitimized the use of calculators in the mathematics classrooms of the schools in those states that have such tests (Long, Reys, and Osterlind 1989). According to a 1991 survey of large-scale testing programs, fifteen of the thirty-five states responding indicated that students were allowed to use calculators on their statewide mathematics tests (Roeber 1991). Connecticut, Missouri, Minnesota, and Michigan are among the states that have allowed calculators to be used on their state tests, and table 9.2 illustrates some different ways that calculators are accommodated.

Table 9.2
Statewide Mathematics Assessments

Connecticut

Eighth-Grade Mastery Test, 1987. Grade 8.
Calculators allowed.
Calculators furnished by the state.
Distributed for classroom use in September 1986.
Four-function TI-1100, model II.

Michigan

Essential Mathematics Test, 1991. Grades 4, 7, and 10.
Calculators not allowed on the mental arithmetic, written computation, and calculator literacy portions of the tests.
Calculators allowed on the conceptual and problem-solving portions.
Calculators not furnished by the state.
Any calculator allowed.

Minnesota

Minnesota Educational Assessment Program, 1989. Grades 6, 9, and 11.
Calculators allowed on all three tests.
Calculators not furnished by the state.
Four-function calculator specified.
86% of the participants permitted to use a calculator did report using it.

Missouri

Missouri Mastery and Achievement Tests (MMAT), 1987. Grades 3, 6, 8, and 10.
Calculators allowed for students in grade 7 and above.
Calculators not furnished by the state. Each school district decided whether to furnish calculators or ask students to bring in their own.
Any hand-held calculator was allowed. In 1987, no particular calculator available was thought to offer an advantage.
The student or the test administrator recorded on each test booklet whether a calculator was used during the test.

The Michigan Experience

In 1986, Michigan replaced its outdated basic skills objectives with more comprehensive and essential mathematics objectives. These objectives make the calculator an integral part of a high-quality mathematics program (Payne 1992). Michigan's statewide mathematics test allowed the use of calculators in 1991. Here are some issues that were detailed during the development of the objectives and the tests:

1. *How should the calculator affect the curriculum?* Whole number paper-and-pencil computation should be reduced. Calculators should be used for decimal and percent computation. Computation with common fraction and mixed-number notation should primarily emphasize concepts and should focus on mental computation. The calculator should be routinely available for all instruction. Students should be taught to make appropriate choices—to use mental computation, estimation, paper and pencil, a calculator.

2. *Should calculators be allowed on the Michigan mathematics assessment test?* Yes, but not on those portions of the test that cover mental computation, basic paper-and-pencil computation (including fractions), and calculator literacy.

3. *Who should furnish the calculators?* Either schools or students should assume this responsibility. The state of Michigan is not in a position, financially, to furnish calculators to schools. The state also feels that buying calculators for schools would send the wrong messages—that calculators are just for the tests and that calculators are more important than other instructional materials. Also, a state-supplied calculator might be different from one with which a student is more familiar. Calculators belong in schools for curricular and instructional reasons, not for test reasons. They should be the responsibility of the students and schools.

4. *Should the type of calculator allowed be specified?* No. A simple four-function calculator has ample capability for all the mathematics contained in the Michigan objectives. A calculator with fraction capability would not give an advantage because fraction items are on the portion of the test where the calculator is not permitted.

   Some calculators have certain features like automatic constants, suppression of nonsignificant zeros, three-key memories, and other keystroking conventions that vary from model to model. It is thought to be unwise to restrict the test to certain calculator features. This type of bureaucratic control might have a stifling impact on school mathematics programs.

5. *Should calculator literacy be tested?* This issue has not been completely resolved. A compromise position may result in reducing to a minimum the amount of testing on calculator literacy. Calculator literacy items attempt to measure a student's familiarity with a calculator. For example, students may be asked to recognize the use of the "memory plus" key. These items would be administered on the portion of the test that does not allow the calculator to be present. It is thought by some that this type of test item will provide helpful information regarding a student's understanding of calculators and will also help send the message to schools that calculators belong in the classroom.

   It is to be noted that calculator literacy is also tested, albeit indirectly, when students are asked to use their calculators to perform routine computations or to solve problems. On the multiple-choice items, some of the foils involve the kinds of mistakes a student might make if the calculator was used incorrectly.

Opponents of testing calculator literacy feel that this particular ability, which is important to address in the classroom, is not a relevant part of the mathematics curriculum and should not be allowed to take up valuable test time.

6. *How will test scores be reported and interpreted?* Formerly, in Michigan, each objective was tested with three "identical" test items, and a student attained an objective by scoring two or three of the three items correct. The new tests will not operate in this manner. Scaled scores will be constructed from sets of test items on several content and process categories. When calculators are allowed on a test, there is little assurance that they are actually used. It will be difficult to know the extent to which scores reflect actual calculator use. How then should the scores of supposed calculator users and nonusers be interpreted? Ideally (perhaps naively), the test administrator would ensure that a student has access to a calculator during a test and the student would be accountable for the decision of whether to use it. The use or nonuse would be transparent to the test scores and test-score interpreter.

FUTURE DIRECTIONS

Assessment

Some believe that making calculators available on tests will transform school mathematics (Heid 1988; Hopkins 1992; Greenes and Rigol 1992). It does not seem, however, to be a wise policy to force calculator use in the classroom by merely allowing them on current tests. Allowing calculators to be used on tests that have not been designed for calculator use may cause only confusion. If calculators are not available in some classrooms, students may perform poorly on some items because of a lack of calculator literacy. Curricular reforms involving calculator use will not be fully implemented in classrooms until high-stakes tests not only allow calculator use but also are specifically designed with calculator use in mind. It seems clear that standardized tests of all types—achievement, aptitude, placement—must be revised to become calculator based. Standardized tests have often been viewed as the gatekeepers to curriculum reform. The reliance on multiple-choice items and the emphasis on certain types of narrow procedural-factual recall types of behaviors have had a stultifying effect on curriculum. Merely allowing calculators on existing tests may not necessarily open the floodgates of desirable reform in mathematics education. Both the curriculum and the tests have to achieve fundamental reforms in partnership.

Some reformers call for the calculator to be available in all testing situations which would put the student in the position of making an appropriate choice of when to use the calculator. This situation may indeed happen some day in the future when the calculator has been thoroughly integrated into the curriculum. In the near future, however, it may be necessary to test some mathematical abilities in a restricted fashion—not allowing calculators for some tests or on portions of some tests—for example, on mental computation.

The presenters at the College Board/MAA joint symposium have given some excellent guidance to the job of redesigning standardized tests so that these tests both permit calculator use and also reflect a calculator-based curriculum. Many questions need extensive research and debate. Some of the following questions have been adapted from John Harvey's list (College Entrance Examination Board 1989).

1. To what extent does using a calculator on a mathematics test item change the nature of what is being measured?
2. How do we know what we are measuring? (Can we be sure that a student did actually use a calculator?)
3. How is the capability of the type of calculator used related to performance on the test item?
4. What effects does calculator literacy have on determining a student's mathematics ability on a calculator-based test?
5. Will errors involving calculator literacy prove to be effective foils?
6. What type of multiple-choice test item is appropriate for calculator use? Some traditional items seem to assume that the student does not have the mathematical power or speed to try out each foil in an attempt to deduce the correct response. A calculator might tempt a student to try to eliminate foils.
7. Will calculator use make the multiple-choice format even less desirable for testing?

The specific calculator used on an examination is of major concern, especially as more capable calculators continue to be manufactured. An item on today's test may require a complicated process or reasoning ability, but with tomorrow's calculator it could be solved with the push of a button. Testing a particular mathematical ability that is entirely subsumed by the push of a key would seem to be a waste of time and effort—for example, asking for the calculation of the square root of 237 to the nearest hundredth. Test developers and curriculum developers will have to stay abreast of new calculators as they continue to produce tests that are aligned with the curriculum and that have validity and reliability. Greater calculator capability and continual student training in specific calculator procedures may become regular aspects of future instruction in mathematics. To legislate that a specific type of calculator be used (or not used) on certain major tests could have a negative impact on instruction.

Curriculum and Instruction

It is difficult to know whether the calculator will affect the secondary school level more extensively than the elementary school level. The use of the graphing calculator seems capable of bringing about major changes in the way we teach secondary school mathematics. The fundamental impact of the calculator seems to be to reduce or eliminate much of the routine manipulative material from school mathematics, thus allowing more time for teaching complex thinking skills and allowing more room in the curriculum to teach geometry, probability, and statistics. Some algebra topics may receive earlier and more widespread instruction. Some topics may be approached in a different manner—for example, graphing functions before solving equations. Certainly, students will have more mathematical power at their fingertips, but how much different will the mathematics be? Will the rate of coverage be faster? Will the depth of coverage be deeper for more students? Computational and manipulative procedures have had such a stranglehold on the curriculum for so long that we find it difficult to contemplate the curriculum with their demise.

Certainly the calculator cannot be allowed to continue as an isolated "add on" in the curriculum (to check homework). Obsolete material will have to be deleted and new emphasis will have to be given to some material (e.g., number sense) to better support routine calculator use. Students will have to be taught strategies for using the calculator to solve problems (e.g., guess and check). Some topics may be taught more effectively with the reinforcement provided by calculators (e.g., the relationship between common fractions and decimals or the relationship between m or b and the graph of $y = mx + b$).

As we consider which traditional pieces of mathematics to delete from the curriculum because using the calculator can replace the need for the skill, we need to become more concerned about any psychological relationships between procedure learning and concept learning (Coxford 1985). There may be some beneficial side effects from mastering some written or mental procedures threatened with removal from the curriculum because they can be performed on a calculator (Hiebert 1990).

On the surface it would seem that the long-division algorithm could be immediately dropped from the curriculum without any necessary compensatory action designed to control for the loss of any important learning. The concepts of division, estimation of division, properties of division, and place value all would seem safe. However, if a fraction-capable calculator would take over and if paper-and-pencil procedures with fractions were not attained, would steps need to be taken to maintain vital skill and understanding that might be lost in related areas of instruction?

It is interesting to speculate on whether or not the future will find a market for specialty calculators that both perform as computational tools or symbol manipulators and are also programmed to give instructional feedback. The TI-12 Math Explorer may be a forerunner of this type of calculator.

The future is not very clear. Certainly, calculators will continue to gain acceptance in schools. Some people will seek to minimize their impact on the curriculum by attempts to regulate their use. Others will begin to embrace the calculator with few or no restrictions. We wish this latter group well.

10

Calculators and School Mathematics

Grayson H. Wheatley

ALTHOUGH inexpensive calculators have been available for more than a dozen years and in spite of repeated calls for their utilization, few elementary school teachers use them (Reys et al. 1990). The use of calculators in elementary school mathematics cannot be considered without addressing issues related to teaching paper-and-pencil computation. What level of paper-and-pencil computational proficiency is appropriate in today's society? For example, how important is it that students be able to divide 3495 by 531 when this operation can be executed using a calculator in a few seconds? By what process do students decide whether to use a calculator, estimate, compute mentally, or use paper and pencil? These and other issues are explored in this paper.

CALCULATOR EFFECTS

What Do We Know about Calculator Effects?

Since inexpensive calculators became available in the late 1970s, many studies have attempted to determine the effect of calculator availability and use on mathematics learning. Educators have explored the role of calculators in instruction. Publishers have even considered building a calculator into mathematics books. What do we know from this research?

A central issue is the effect of calculator availability on computational proficiency. The fear that calculators will undermine students' ability to perform paper-and-pencil algorithms seems unfounded. Beginning in 1977, Shumway and his colleagues conducted a year-long study of calculator use in elementary schools. At five midwestern sites, calculators were used extensively in grades 2–6 for one year. Results show that computational scores on standardized tests did not decline (Shumway et al. 1981; Wheatley et al. 1979). In a meta-analysis of calculator studies, Hembree and Dessart (1986, 1992) confirmed this finding and reported that in many cases computational scores actually improved.

Regarding problem-solving performance, Hembree and Dessart's meta-analysis showed that scores were higher when calculators were used. Students were more effective in selecting and using problem-solving methods, and they were more confident, enthusiastic, and persistent (Hembree and Dessart 1986, 1992; Reys and Reys 1987; Szetela and Super 1987; Wheatley and Wheatley 1982).

An important issue in mathematics learning is persistence—working at mathematics tasks for extended periods of time until a solution is reached. Silver reports (personal communication) that students in conventional instruction expect to solve a problem in thirty seconds. Such a belief can be debilitating because longer periods are often required to become meaningfully engaged in problem solving. However, there is evidence that when students are engaged in problem solving with calculators, they become more persistent. The calculator appears to play a significant role in this change (Wheatley and Wheatley 1982).

The Tool Metaphor

Throughout history, humans have invented and used tools. From stones fashioned for scraping skins to sophisticated telemetry devices for tracking spacecraft, tools have been instrumental in cultural advances. A calculator is simply another tool that has been created to facilitate our activities.

Calculators threaten the existence of the paper-and-pencil algorithms that are so firmly entrenched in our schools, and so they are viewed with alarm. Teachers, parents, and administrators fear that children will not learn mathematics if calculators are available in the classroom. However, insisting on paper-and-pencil computations is like requiring people to ride a bicycle from city to city when trains and automobiles are available. Because calculators are a powerful alternative to the drudgery and inefficiency of paper-and-pencil arithmetical computation, arguments for traditional mathematics education ring hollow. Practicing poorly understood paper-and-pencil procedures may even militate against learning to think mathematically.

The Calculator as a Computational Tool

Although a calculator is often viewed as a tool for carrying out complex computations once a plan is formulated, calculators can be quite useful in deciding how to solve a problem. In the act of devising a problem-solving plan with a calculator, good problem solvers will often perform a variety of exploratory moves, trying certain computations in the process of getting to know the problem. Consider the problem, "Notebooks cost $1.57 each. How many notebooks can Pat buy for $40.00

and how much change will she receive?" I have observed grade 6 pupils using exploratory moves such as multiplying 10 × $1.57 then 15 × $1.57, and then 22 × $1.57 in the process of deciding how to solve the problem. These exploratory moves were greatly facilitated by calculators. Such exploratory methods were not observed as frequently when students did not have a how-to-proceed scheme that included calculators. When thinking of doing the computations "by hand," many students would instead do nothing. However, the calculator-experienced students readily tried a variety of computations in their attempt to understand the problem and to devise a plan (Wheatley and Wheatley 1982). In this way, calculators freed students to reason mathematically without being detoured by time-consuming paper-and-pencil algorithms.

BELIEFS

There is strong evidence that calculator use by elementary school students facilitates mathematics learning (Hembree and Dessart 1986; Shumway et al. 1981; Wheatley and Shumway 1992; Wheatley and Wheatley 1982). However, many teachers, administrators, and parents believe that putting calculators in the hands of young children is undesirable because it will result in a nation of nonthinking button pushers. For example, Coburn (1990) and Reys et al. (1990), surveying educators in Michigan and Missouri, reported that calculators are used to some extent in fewer than one-half of the elementary school classrooms and not at all in one-third of the classrooms. Although it is encouraging that Reys et al. (1990) found an increase in calculator use during the 1980s, the resistance is still strong. This resistance is based on certain assumptions, which should be examined.

One assumption is that mathematics is a set of computational procedures involving memorizing facts and computing with numbers. This archaic view contrasts sharply with the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) and with a constructivist approach, in which mathematics is the activity of constructing patterns and relationships.

Second, when calculator use is contemplated, it is assumed that students will continue with the same drill-and-practice worksheets based on behavioristic learning theory. However, there is resistance to the constructivist approach, which does not use drill and practice.

Computers vs. Calculators

Ironically, computers are acceptable in mathematics classes, but calculators are not (Williams 1983). Because we are still in the exploratory phase of using computers in education, judgment is being suspended while we await study results. In general, however, computers are thought to have great potential for transforming society. Their aura suggests great power.

Calculators, however, have lost their appeal. They are "the previous technology." Calculators compute—something traditionalists believe students should do themselves. So, bringing them into the school classroom attacks one of the three Rs.

Myths

Certain myths have formed about the detriments of calculator use in schools. The sooner these myths are examined and debunked, the sooner calculators will find their way into mathematics classrooms:

Myth 1. Children should learn the facts and computational procedures *before* being allowed to use calculators. Supporting argument: The use of a calculator will undermine mathematics learning because mathematics is knowing facts and computational procedures.

However, the architectural metaphor that makes facts and skills the foundation stones of mathematics does not apply to school mathematics of the 1990s (NCTM 1989).

Myth 2. If you let children use calculators, they will not learn mathematics—they will just be pushing buttons. Supporting argument: Mathematics is a set of specific procedures.

However, most mathematics educators view mathematics as a network of meanings in which procedures play a relatively minor role.

Myth 3. Calculators will destroy students' ability to think. Supporting argument: Using a calculator is the mindless entering of meaningless numbers.

Myth 4. Calculators should be used to check paper-and-pencil computations. Supporting argument: Calculators will "reinforce" paper-and-pencil methods. This argument is based on a belief that mathematics is skills and that these skills are learned through reinforcement.

ARITHMETICAL COMPUTATIONAL PROCEDURES

Three questions loom large when considering the place of computational procedures in school mathematics:

1. Is it important that students be able to carry out complex arithmetic computations with paper and pencil?

2. In what way, if any, does learning an algorithm imposed by the teacher contribute to mathematical knowledge?
3. What mathematics is fundamental for the twenty-first century?

The school mathematics curriculum has its present form and orientation not because teachers think that it is the way to teach mathematics but because of tradition. With calculators available, a redesigned curriculum is needed. Although teachers expect algebra students to know certain rote methods, such competencies may not be essential and, in fact, have a negative effect on students' mathematics learning. For example, if the majority of students' time in mathematics class is spent learning procedures, they may come to believe that computation is mathematics and be less inclined to construct mathematical relationships. Even so, middle school teachers may not use calculators because they are attempting to develop the "skills" that they believe are required for algebra. Calculators weaken this argument considerably.

Alternatives to Imposed Arithmetic Algorithms

The procedure of choice for numerical computing is shaped by the society in which we live. For example, when people sat on high stools and performed arithmetical computations with a quill pen, a knowledge of procedures for determining products and quotients was valuable and thus became an essential part of every student's education. Paper-and-pencil arithmetical computational procedures became institutionalized in schools during the nineteenth and early twentieth centuries because they were needed in that era to carry out numerical operations. Today, tedious computations are frequently performed by calculators and computers, often with a minimum of direction by the user. Special-purpose cash registers have icons rather than numerals along with a button labeled TOTAL, so that the salesperson has no need to compute. Although it is not unusual for archaic practices to be taught in schools decades beyond their utility, the advent of calculators and computers has made standard paper-and-pencil computations inappropriate.

Recently, researchers have developed models of children's number constructions to help examine educational practices and design instructional activities (Steffe and Cobb 1988). Constructivists believe (von Glasersfeld 1987) that children should be encouraged to construct their own computational methods rather than having standard algorithms imposed on them (Cobb et al. 1990). When students construct mathematical relationships, they see computational procedures as an integral part of mathematical conceptualization. Here's an example of a self-generated algorithm for adding 39 and 53 in vertical format: "Take 1 from the 3 to make 10 (9 + 1), 10 and 80 (30 + 50) is 90, 92 (2 left from the 3)."

Problem-centered Learning (Wheatley 1991) stresses social interaction and the sharing of solution methods. Through interaction, children can construct computational algorithms while building mathematical relationships. There is strong evidence that second graders who have been encouraged to construct their own methods show greater mathematical competence, and the superiority is lasting (Cobb et al. 1991).

It is not at all clear that standard paper-and-pencil algorithm instruction contributes to relational understanding (Skemp 1979). Although we can attempt to discuss whether children understand the arithmetical algorithms they study (*instrumental understanding*, using Skemp's term), little evidence is available that such "understanding" contributes to mathematical meaning making. In fact, learning and practicing the long division algorithm may lead to beliefs that "understanding" a procedure is understanding mathematics (Wheatley 1980). Remembering a set of steps for computing is the opposite of constructing number relationships.

It is indeed more likely that calculator-enhanced activities that help students construct mathematical relationships (illustrated below) would be more appropriate and effective. As students construct their schemes for computing, a variety of alternatives would be viable, depending on the setting. Listed below are five alternatives:

1. Mental arithmetic. Children who have learned mathematics meaningfully can easily and quickly determine the sum of 385 and 120 using thinking strategies (e. g., 385 + 115 is 500 and 5 more gives 505).
2. Estimation. For many purposes, an estimate rather than an exact answer is more useful (e.g., with large numbers thinking of 23 or 24 billion instead of than 23 573 591 207).
3. A self-generated algorithm.
4. A calculator.
5. Computer software. For example, spreadsheets, where the computation is performed automatically, are a practical and a powerful tool.

Deciding on a Method

When calculators are introduced into classrooms, students do not spontaneously use them, even for complex computations. Students must learn to construct action schemes that include calculators as an option. For example, sixth graders without extensive calculator experience have been observed performing long division by paper and pencil when a calculator was lying on their desk (Wheatley and Wheatley 1982). In some situations, students rejected calculators because they could not understand the decimal display. It was clear they had been expecting a whole number and possibly a remainder, in paper-and-pencil style, and were unprepared for the decimal that appeared.

Thought Experiments

When people decide to use a calculator, they often perform a *thought experiment* (Lakatos 1976). We all use thought experiments (TE's) many times a day. For example, in deciding which route to take from Tallahassee to Orlando, I imagine driving Interstates 10 and 75 and compare that experience to taking U.S. 27 through small towns and the countryside. It is on the basis of these vicarious experiences that I decide which route to take.

When deciding how to carry out an arithmetical operation, an individual often constructs an anticipated sequence of moves and "runs through" the activity mentally before actually entering numbers. For example, I might want to know the probability of winning the lottery if I buy $50 worth of tickets for 524 days in a row when the probability of winning for each ticket is 1/575 757. I might recognize that the computations involved in answering this question will require multiplying and dividing large numbers and decide that a calculator would be better than using paper-and-pencil methods. Such TE's allow an individual to explore and evaluate the efficacy of using a calculator in comparison to other possible methods. The act of performing such thought experiments can provide the basis for decision making. As one performs a TE and reaches a decision, a repertoire of experiences is built that forms a general decision-making scheme. When, in the future, similar decisions are to be made, the results of the previous thought experiments are integrated into a decision-making scheme that can be used without repeating the TE at that time—perhaps some new TE will be conducted using previous experiences. From the context of the task, the question becomes, Should I perform this calculation mentally? Estimate? Use paper and pencil? Use a calculator? Discuss it with someone? Use a computer? In order to become powerful users of calculators, students must perform such TE's and build how-to-proceed schemes.

To perform calculator TE's, students need to have opportunities to engage in "calculator thinking." One must have performed operations with a calculator in order to know what it is like. Such experiences can form the basis for imagining what it would be like to perform a series of arithmetical computations in each of several ways. When faced with the need to perform calculations, students familiar with TE's can make informed decisions.

CALCULATORS, NUMBER SENSE, AND MEANING

Calculators can help students develop number sense. With a well-developed number sense, students can construct computational procedures while focusing on the meaning of the mathematics. When learning mathematics is meaningful to students, mental arithmetic becomes an inherent part of doing mathematics, and the differences between conceptual and procedural knowledge become less distinct (Hiebert 1986).

Number sense is a major theme of the NCTM *Curriculum and Evaluation Standards*. It is important that students give meaning to mathematical ideas, that they construct the objects of their mathematical world. Number sense depends on an elaborate network of interrelated schemes constructed as conceptual operations. For example, having constructed ten as an iterable unit (Cobb and Wheatley 1989) allows a person easily to determine ten more than 73. A child who can respond confidently to such questions has constructed ten as an iterable unit, a major construction essential in building number sense. Calculators can contribute to building number sense as illustrated by the following activity.

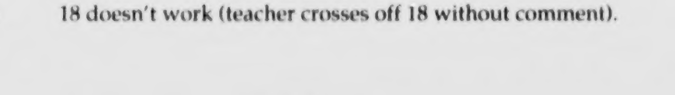
A Calculator-enhanced Mathematical Activity

Calculators can be most helpful in constructing number relationships. Consider the range game (Wheatley 1990) as an example. The class is asked to enter 75 in a calculator and choose a number that when added to 75 gives a number in the range from 90 to 110. As possible numbers are suggested by the class, the teacher writes on the board whatever numbers the students report. Questions about the smallest or largest number naturally arise in this discussion, and soon the list of whole numbers from 15 to 35 is obtained. Following is the dialogue that resulted when the range game was played in a grade 6 class:

Teacher: Find a number that when multiplied by 37 gives a number in the range from 500 to 600.

The teacher writes numbers suggested by students on the board.

37 × ____ = ____



S: 15.

P: 18.

F: 14.

W: 18 doesn't work (teacher crosses off 18 without comment).

M: 16 is the largest number.

Teacher: Why do you say 16 is the largest?

M: Because I tried 17 in my calculator and it was too big.

E: 14 is the smallest.

Teacher: Do we have them all now?

Class: Yes.

A: Wait a minute; 14 is *not* the smallest. I tried 13.9 on my calculator, and it works (75 × 13.9 = 514.3).

This unexpected response launched the class into a fervent inquiry for more "numbers like that."

The range game embodies computational estimation, decimals, and limits and in general promotes number sense. With computations performed on a calculator, the student can focus on number relationships. There are many other activities where the calculator plays an instrumental role in developing mathematical reasoning and concepts (Reys et al. 1979).

Guidelines for Calculator Use

A calculator should be used when it—

1. contributes to constructions and reconstructions;
2. permits attention to be given to meaning;
3. promotes mental arithmetic and estimation;
4. creates problematic situations;
5. promotes number sense;
6. facilitates problem solving;
7. allows learners to consider more complex tasks;
8. facilitates a search for patterns;
9. removes the anxiety of computational failures;
10. provides motivation and confidence.

RESEARCH QUESTIONS

How do calculators facilitate scheme building and conceptual elaboration? The role of calculators in the construction of mathematical meanings is unexplored. Teaching experiments designed to study this question could prove quite enlightening.

What is it like to construct mathematical meanings in a calculator-enriched learning environment? Previous research suggests that the use of calculators allows students to attend to mathematical relationships rather than poorly understood computational procedures. It would be useful to have models of children's mathematical actions when they construct meaning using calculators as a tool. Further, more powerful graphing calculators, which are now used in high schools, may be particularly effective in helping young children form mathematical relationships. Much can be learned by studying the use of these devices with students in the intermediate grades.

How does calculator use facilitate the construction of multiplicative structures? The inherent difficulty of subtraction and division in relation to addition and multiplication is well documented. By freeing learners to focus on relationships rather than computational procedures, calculators will perhaps encourage the building of action schemes that include multiplication and division as part of the same scheme so that children may flexibly use subtraction in missing-addend situations.

To what extent do calculators encourage students to reflect on their activity (Wheatley 1992)? All too infrequently students distance themselves from their activity and reflect on what they are doing. Students may have knowledge in action but be unable to look down on their activity. Piaget (1970) has identified reflective abstraction as a major mechanism of learning. Using calculators may promote reflective abstraction in ways not yet understood.

SUMMARY

In this paper, the intertwined issues of calculator use and appropriate computational procedures are considered. In spite of the widespread view that calculators could undermine mathematics learning, no evidence to support such a position exists. On the contrary, considerable research evidence suggests that calculators can play an important role in children's mathematics learning. In a learning environment in which calculators are available, the emphasis can be on meaningfulness rather than computational procedures.

Calculators should take their place with mental arithmetic, estimation, and paper-and-pencil methods in the school mathematics curriculum. Calculators can provide rich opportunities for mathematics learning, as the examples in this paper show. Although certain issues about calculator use are clear, many unanswered questions remain. Research is needed to guide educational decision makers.

11

Overview and Proposal on the Use of Calculators in Japanese Schools

Hanako Senuma

PHILOSOPHICALLY, this paper considers the calculator a tool to promote mathematical activities. That is, calculators are not a substitute for mental calculations and paper-and-pencil methods or merely a tool for enrichment. Rather, calculators are a useful tool for widening the scope of mathematics education. Calculator use in Japan must eventually address many issues, including their role in computation and their relation to entrance examinations.

This paper presents an overview of trends in calculator use in Japan and discusses why calculators are not widely used in Japanese mathematics classrooms. Five issues related to calculator use in learning are also discussed.

WHAT HAS HAPPENED?

An examination of calculator-related activities in mathematics education in Japan during the last ten years reveals that calculators are affecting school mathematics programs, but the changes are taking place slowly.

Journals

A review of calculator-related articles published during the last ten years in ten professional journals from three Japanese societies—JSME (Japan Society of Mathematical Education), JSSE (Japan Society for Science Education), and SEME (Society of Elementary Mathematics Education)—as well as two commercial journals by Meiji Tosyo Company reveals that only a few such articles were published in each journal each year. Although the number of calculators increased in society, the number of mathematics education articles on calculators did not increase. However, several special issues of some journals have been devoted to calculators, including the following:

I wish to acknowledge the advice and help of Eizo Nagasaki of the National Institute for Educational Research in preparing this paper.

1991 *Arithmetic Education* (Itoh et al. 1991)
"Special Issues: How Should We Use Calculators in Arithmetic Lessons?"

1990 *Elementary Mathematics* Teaching Today (Machida et al. 1990)
"Special Issues, Part 2: How Should We Use Calculators?"

1982 *Arithmetic Education* (Sueyoshi et al. 1982)
"Special Issues: How Should We Do Computational Revolution by Calculators?"

Reports

Notable reports on calculators in Japan were Katagiri's (1983), which focused on the elementary school level (grades 1–6), and Nagasaki's (1989), which focused on the lower secondary school level (grades 7–9). Table 11.1 summarizes reports and projects about calculators and also shows the percent of schools equipped with computers. Since 1984 the Japanese Ministry of Education has surveyed schools about computers but not calculators.

Table 11.1
Reports about Calculators and Percent of Schools Equipped with Computers

			Schools with Computers (%)		
			Grade		
	Projects	Reports about Japan	1–6	7–9	10–12
1991			41	74	99
1990			31	59	98
1989	Nagasaki's report		21	45	96
1988		England report after visiting Japan	14	36	94
1987		Nagasaki's report for UNESCO meeting	7	23	86
1986			2	13	81
1985			—	—	—
1984			1	3	56
1983	Katagiri's report		—	—	—
1982			—	—	—
1981			—	—	—
1980			—	—	—
1979	Shimada's report for IEA		—	—	—

Comparison with Other Countries

The Second International Mathematics Study reported that calculators were used much less in Japan than in other countries. For example, only 3 percent of Japanese seventh-grade classrooms used calculators with four basic operations, which was the second lowest of nineteen countries.

In grade 12, on calculators with scientific functions, Japan was the lowest of fifteen countries with a rate of 5 percent (National Institute for Educational Research 1986).

After attending the meeting of the UNESCO pilot project, Nagasaki compared four countries and concluded, "The countries where calculators are allowed and used in the mathematics curriculum are Australia and New Zealand. The country where calculators are allowed in the mathematics curriculum but not used is Japan. The country where calculators are not used for economic reasons is Pakistan" (Nagasaki 1987a, 1987b).

Some members of the Mathematics Working Group for National Curriculum in England visited Japan and had the following impressions: "In view of Japan's reputation as one of the most technologically advanced nations of the world, it came as a great surprise to find that there is relatively little calculator and computer provision in schools" (Department of Education and Science 1988).

WHAT IS HAPPENING?

The Role of Calculators

Calculators play two roles in mathematics education (Shimada 1985). One role is as "glasses for short sight," namely, an aid for computation, covering weak points in paper-and-pencil methods. The other role is as "a magnifier or a microscope," namely, a tool for discovery, providing rich data that cannot be obtained with paper and pencil only.

Engel (1980) listed fifteen issues in mathematics education that arose from the introduction of calculators. In relation to this paper, (the third issue) is most important. Engel says that even though paper-and-pencil computational proficiency with multiplication by one or two digits may be enough to live in a society that has calculators available, we should not drop completely the paper-and-pencil multiplication algorithms from the curriculum.

The idea that it is more important for a child to be able to compute using his or her own method than to master a definitive form of paper-and-pencil method was expressed by Shimada, Katagiri, and Hirota (1981). With the availability of calculators, children should be encouraged to construct their own computational methods rather than be forced to learn a fixed procedure imposed by the teacher.

The Use of Calculators in Schools

A survey of calculator use by elementary and lower secondary school teachers in Tokyo was reported by Nagasaki and Senuma (1989). This research reported responses from 58 elementary schools (29 percent reply rate) and 111 lower secondary schools (56 percent reply rate). The average number of calculators for mathematics teaching was ten per school in elementary schools and nine per school in lower secondary schools. Table 11.2 shows that 60 percent of the elementary schools and 48 percent of lower secondary schools reported that no calculators were available. It was also reported that calculators were not used in nearly 60 percent of both schools.

Table 11.2
Total Number of Calculators per School in 1988–89

Number of Calculators	0	< 10	< 20	< 50	< 100	100 ≤
Elementary	60%	14%	7%	16%	1%	0%
Lower Secondary	48%	22%	16%	13%	2%	0%

Two main reasons why calculators were not used in elementary schools were (1) concern that the use of calculators would reduce the students' ability to compute (reported by 45 percent), and (2) an insufficient number of calculators were available (reported by 16 percent). At the lower secondary schools, four reasons were identified: (1) they reduce the ability to compute (28 percent); (2) we have no content that requires calculators (22 percent); (3) we have no time to use calculators (11 percent); and (4) they reduce the ability to think (10 percent).

Some research revealed that the opinions of mathematics teachers toward calculators differ from those of university professors, with professors tending to support calculator use. For example, in a survey of teachers of public lower secondary schools, teachers of national lower secondary schools, supervisors, and university professors, Nagasaki (1988) reported that although about 40 percent of each group supported the use of *computers* in school, fewer than 20 percent of the teachers agreed with the use of *calculators* compared to 54 percent of university professors. A similar situation is seen in Katagiri's (1983) survey of parents, elementary school teachers, and university professors, where 17 percent of parents, 49 percent of teachers, and 83 percent of university professors are positive toward calculator use in mathematics classrooms. Thus there is a significant gap between any two of these three groups.

A longitudinal study on students' mathematics achievement and attitude from elementary to upper secondary school (Senuma et al. 1992) revealed that non-calculator use in mathematics lessons increased from grade 7, as shown in table 11.3.

Table 11.3
Percent of Students Who Selected the Item "Calculators Are Rarely Used in Mathematics Lessons"

Grade	5	6	7	8	9	10	11	12
Percent	78	78	95	93	91	98	93	92

In the same study, another item concerning calculators was included. The choices and student responses are reported in table 11.4.

Table 11.4
Students' Views on Calculators and Mathematics

Response	a	b	c	d	e
Grade 6	29%	6%	51%	6%	8%
Grade 7	25%	5%	47%	9%	14%
Grade 9	19%	2%	49%	11%	19%
Grade 10	10%	3%	44%	18%	25%
Grade 12	9%	2%	46%	17%	26%

a. Calculators shouldn't be used in mathematics lessons because they reduce the ability to compute.
b. Calculators should be used to solve every problem.
c. Calculators may be used to solve problems with complex numerical values.
d. Calculators should be used exclusively to solve problems with complex numerical values.
e. There is no relation between calculator use and mathematics lessons.

Table 11.4 also suggests that students' fear that their computational ability will decline from the use of calculators decreases as they progress to higher grades. Another survey revealed that although students have experiences with calculators at school or home, most are not concerned that calculators are prohibited on tests (National Institute for Educational Research 1992).

If calculators are to be used in mathematics lessons, it is essential that students be able to interpret the results of a calculator display. For example, a calculator may provide many more digits than are needed for a meaningful solution. The item in figure 11.1 was answered correctly by 47 percent

Following are data about the height of nine seventh-grade male students:

148.3	145.6	147.2
138.9	144.5	152.7
146.3	145.1	147.8

After calculating the mean value, the calculator displays 146.2666666.
What is the most suitable value as a mean of height?

a. 146 cm b. 146.3 cm c. 146.27 cm d. 146.2667 cm e. 146.2666666 cm

Fig. 11.1. Item used in assessing understanding of the results of computation

of students in fifth grade, 59 percent in sixth grade, 73 percent in seventh grade and 83 percent in eighth grade (Sawada et al. 1991).

In order to obtain accurate results, data must be entered correctly on a calculator. In an effort to determine the frequency of button-pushing errors, the problem in figure 11.2 was given to students in grades 7, 8, and 9 in lower secondary schools (Nagasaki 1989).

The following numbers show the height in (cm) of thirty-nine boys' in a junior high school class. Please calculate their average height by using your hand-held calculator.

1. Add all numbers

154	156	152	157	138
146	146	150	142	157
131	155	149	157	138
144	143	146	139	147
163	149	137	155	149
149	144	149	152	157
145	136	149	140	166
156	140	160	148	

Total: ____ cm.

2. Calculate the average height of the thirty-nine boys: ____ cm.

3. Did you push a key(s) by mistake?

a. Yes b. No

4. In what direction did you add the numbers?

a. Left to right b. Top to bottom c. Don't know

5. Have you ever used a hand-held calculator in mathematics class?

a. Yes b. No c. Don't know

Fig. 11. 2. An example of coping with many data

Although the percent of correct responses increased by grade level, this research documents that there were many errors on these problems. Most errors were caused by using the wrong keys. More specifically, about 40 percent of the students in each grade pushed one or more incorrect keys. This suggests that strategies for recognizing reasonable answers are important when using calculators.

The Mathematics Curriculum

Using computational tools was mentioned in the *Curriculum of Mathematics for the Elementary School* as early as 1968, but the specific mention of calculators first appeared only in 1989. Table 11.5 highlights the recommended treatment of computational tools over a thirty-year period.

Table 11.5
Term and Treatment Related to Calculators in a Mathematics Course of Study

Year		Elementary School	Lower Secondary School	Upper Secondary School
1989	A	1992	1993	1994
(E-L-U)	B	Dentaku	Dentaku	Dentaku
	C	5th and after	all grades	in optional courses
1977 (E-L)	A	1980	1981	1982
	B	Keisanki	Keisanki	Keisanki
1981 (U)	C	5th and after	all grades	in Mathematics II
1968 (E)	A	1971	1972	1973
1969 (L)	B	Keisanki	Keisanki	Keisanki
1970 (U)	C	4th and after	all grades	General Math, Math IIa, Applied Math
1958 (E-L)	A	1961	1962	1963
1960 (U)	B	no description (N. D.)	N. D.	N. D.
	C	N. D.	N. D.	N. D.

Year: The year that each course of study was announced.
(E: elementary; L: lower secondary; U: upper secondary)
A: The year each curriculum was implemented.
B: Japanese term related to calculator that was used in the course of study.
C: Grade or subject name that "Dentaku" or "Keisanki" was recommended to use.

In the curricula of 1977 and 1968, the term *Keisanki* was used. This term means "computational tool" and referred to computers, calculators, and other computational tools. Calculators were not the main focus of this term, because it related to algorithms or other content. In 1989, *Dentaku* appeared in the *Curriculum of Mathematics for the Elementary School, Lower Secondary School, and Upper Secondary School* and directly referred to hand-held calculators. By this appearance, the use of calculators in teaching computation is officially recommended.

The current statement for calculators is as follows (Nagasaki, Sawada, and Senuma 1990):

Elementary School (Grades 1–6)

> At the fifth grade or later, the teacher should help children adequately use soroban and calculators, to lighten their computation burden, to improve the effectiveness of teaching in situations where many large numbers are to be processed (e.g., statistics), or where pupils confirm whether the laws of computation hold in the multiplication and division of decimal fractions. At that time, the teacher should provide adequate situations in which the results of computation may be estimated and computation may be checked through rough estimation.

Lower Secondary School (Grades 7–9)

> In the teaching of numerical computation, the teacher should give consideration to improve the effectiveness of learning by having the students use soroban or hand-held calculators, etc. as an occasion demands.

Upper Secondary School (Grades 10–12)

In the teaching of computation, the teacher should have students use calculators and computers as an occasion demands, so as to improve the effectiveness of learning.

FACTORS HINDERING THE USE OF CALCULATORS IN JAPAN

Language

Numerals and number facts in Japanese are easy to remember. The Japanese numerical system is based on the decimal system. We have traditional songs of basic number facts, and many people say they are easy to remember. For example, by the end of second grade, almost all pupils can recite basic multiplication facts. Mental computation makes calculator use unnecessary.

Textbooks

Japanese mathematics textbooks contain about 200 pages and include the minimum content that all students should master. Textbook writers and editors are rather conservative about introducing calculators into textbooks. Although the current curriculum allows for calculator use above grade 5 and some new textbooks have several pages on calculators, most textbooks have been slow to integrate calculator usage.

Philosophy of Education

A Japanese education has a theoretically oriented nature rather than a pragmatic one. In fact, school and society are considered to be different and unconnected. Therefore, the importance of calculator use in society does not directly influence calculator use in school.

Teachers

When students are not able to learn something, many Japanese teachers think it is caused by their lack of teaching ability. For example, when students show lack of ability with paper-and-pencil computation or with mental computation, they think they have to teach it more perfectly rather than to complement this lack with the use of calculators. If the calculator becomes an integral part of mathematics programs, the teachers will think they have to teach both.

How Mathematics Is Treated

Teachers of mathematics prefer to teach theory of mathematics rather than applications. This tendency becomes stronger as the school level becomes higher. At the elementary school level, understanding number concepts and operations is one focus, whereas at the secondary school level, the learning of theoretical mathematics is the focus. This will result in elementary schools using calculators more than secondary schools.

Areas of Emphasis

Japan stresses both the answer and describing how the answer was found. For example, in algebra the importance of writing down both the answer and the mathematical expression and equations is traditionally stressed. This approach helps students to set up and eventually solve the problem, and it also provides a record of how students think. Also, in the Japanese curriculum, writing both an answer and an explanation is considered an important way of communicating in mathematics.

Established System of Teaching Computation

Our system of teaching paper-and-pencil computation and mental computation have been established very well, and many mathematics educators want to hold onto this well-organized system. In addition, student computational achievement is high.

FUTURE DIRECTIONS

The present level of calculator use in Japan is low. This does not mean that calculators are not important, since many Japanese mathematics educators believe that the desired goals of mathematics education may be achieved through the use of calculators. Several issues remain to be resolved before calculators are used more extensively in the Japanese mathematics curriculum. Five of these issues will be discussed.

Early Use of Calculators

In elementary school mathematics, understanding place value and mastering simple skills of the four operations with whole numbers are important objectives, and the early introduction of calculators will be examined from this viewpoint.

First, calculators should not be used for teaching an initial understanding of the place-value numeration system, because the student cannot see computation process. A clear image of the process of computation is very helpful for understanding number relationships at the elementary school level. Understanding place value is facilitated by *soroban* methods, mental calculations, and paper-and-pencil methods, but place value is not as easily understood when calculators are used.

Second, it is relatively easy for Japanese children to master the skills of the four operations with whole numbers by mental calculation and paper-and-pencil methods without using calculators. As mentioned earlier, the Japanese language makes it easier for students to understand place value because ten is written first for the teen numbers. Therefore, I do not believe that calculators should be introduced in lower grades to students who are unable to compute mentally. If children are unable to compute in these grades, it can be said that they do not completely understand the place-value system. Therefore, if calculators are used as a substitute for mental calculation or paper-and-pencil methods at this stage, children may not develop a thorough understanding of place value.

In conclusion, it is not good to introduce calculators in the lower grades. However, calculators should be introduced when children compute with large numbers.

Entrance Examinations

In Japan, calculators have never been used on entrance examinations at any level. Here are some reasons why calculators are not used:

1. The examinations have not included mathematics problems that use calculators.
2. Confusion would be caused by students using different types of calculators.
3. Calculators have typically not been used in schools, and so examiners are not experienced in using calculators.

However, a new direction has emerged. Professor Fujita, the Japanese representative for the International Commission on Mathematical Education, proposed that calculators should be used on upper secondary school university entrance examinations (Fujita and Kaminaga 1986). Furthermore, in the "Meeting for Mathematics Education of Upper Secondary Schools," chaired by Fujita, some members are planning to develop calculator-based problems for university examinations. In Japan, entrance examinations exert strong influence on lower school levels. If calculators were used on university entrance examinations, then calculators might be used in upper secondary schools. Finally, the use of calculators may be accelerated in classroom lessons of lower secondary schools and elementary schools.

Viable Computational Alternatives

Selection among the computational alternatives of mental computation, paper-and-pencil computation, and the use of calculators should be made according to the problem situation. Calculators may be used after mastering paper-and-pencil computation to some extent (for example, 2-digit number + 2-digit number, or 3-digit number × 2-digit number), but not as a substitute for paper-and-pencil computation. Even if calculators are introduced, they should not always be used. It is important that children select appropriately among the alternatives according to the problem situation. Namely, it is important that children be able to solve problems confidently by themselves in some manner.

Shiono, who played a central role in deciding the mathematics curriculum in the Ministry of Education in the 1930s, has advocated mental computation and has expressed a positive attitude toward calculator use. He suggests mental computation might become increasingly important with the use of calculators.

There are differences in the manner in which problems are approached when calculators are used compared to paper-and-pencil computation. In paper-and-pencil methods, attention is focused on number patterns. When calculators are used, the number pattern may not be seen, for example: $1 + 2 + 3 + \ldots + 100 = (1 + 100) + (2 + 99) + \ldots$ for similar problems in this same category, some are appropriate for calculators, but others are more appropriate for paper-and-pencil computation. For example, in pattern-finding problems, the use of a calculator may be efficient for finding a rule of squared numbers: $(67)^2 =$, $(667)^2 =$. However, paper-and-pencil computation may be more efficient than using a calculator for finding a pattern in the addition of a number and its reversely ordered number: $34 + 43 =$. It would be helpful to classify mathematical problems by the following criterion: those that are appropriately solved by using a calculator and those by paper-and-pencil computation.

The use of calculators will increase the number of situations where complex numerical values are treated, which will make estimation more important. Thus when calculators are used, approximate and significant figures become more important (Shimada 1984). The relationship between calculators and estimation needs to be better understood. More research is needed, since calculator and estimation activities are important computation activities.

Problem Solving

If calculators are used in problem solving, the burden of computation can be lightened and children can more easily grasp the meaning of problems. Furthermore, more complex and more realistic numerical values can be treated (Senuma 1989). For example, an interesting discussion of problems involving the area of a polygon was provided by Shimada, Katagiri, and Hirota (1981). What is important in thinking about the area of geometrical figures is deciding which lengths should be measured. In asking "area of polygon" questions, teachers tend to give simple numerical values for lengths (usually whole numbers). Children are required only to substitute the values in a given formula. If calculators were used, teachers could begin their lessons by asking which lengths should be measured and having the students decide on the length to be used.

Several significant problem-solving projects have occurred in Japan. One project involved researchers and teachers developing problems with a variety of solutions (Shimada 1977). Problems with complex numerical values were intentionally avoided so that children could focus on the process of mathematical model making. In other projects, children themselves developed new problems by changing a part of the problem (Takeuchi and Sawada 1984; Nagasaki and Hashimoto 1985). In these projects, when children made problems with complex numerical values, they were not chosen to be used with the whole class. Developers of these projects recognized that the process of selecting information and formulating it was very important in mathematical modeling. However, they had to avoid problems with large numbers because of time restrictions for calculation. With the availability of calculators, this restriction can be removed. As a result, mathematical modeling and mathematical problem solving can be more effectively developed using calculators.

Mathematics Education

There are two main views of mathematics. In one (theory oriented), mathematics is viewed as a collection of theories, and mathematical beauty is in the completeness of those theories. Another view (application oriented) focuses on mathematical situations that can be applied to real problems in social and natural settings.

The former is important for children who will study mathematics (mainly algebra) in universities, in order to foster mathematics thinking. The latter is regarded as the aim for children who will use mathematics in university or society in the future, in order to foster engineering thinking. It seems that the *Curriculum of Mathematics* rather emphasizes the mathematics needed to become a mathematician. As mentioned earlier, teachers of mathematics tend to emphasize theory, and this tendency becomes stronger as the school level increases (Shimada 1979). This is still the main reason why calculators are not as widely used in secondary school as in elementary school (Senuma and Nagasaki 1990). If we are interested in applied mathematics or mathematics for all as the aim of mathematics education, the latter view should be emphasized more in Japanese mathematics education.

Calculators are compatible with both views. For example, in theoretical mathematics calculators can be useful (algebraic approach) in finding patterns that lead to conjectures and proof. Calculators can also be useful in the mathematical modeling of real situations. In this approach, not only can calculators be used for calculation, but also ideas of problem solving can pervade all aspects of class activities (Ohta 1989).

Calculators should not be used as a substitute for mental computation or paper-and-pencil computation but as a means for facilitating problem solving or mathematical activities and for widening their possibilities. If calculators are used in problem-solving or mathematical activities or in fostering abilities to solve problems, then calculators might be used in lower grades of elementary schools. However, in instances where children cannot carry out mental computations or paper-and-pencil computations and children do not understand basic number structure or the basic four operations on which mental computation and paper-and-pencil computation are based, we should not introduce calculators as a substitute for them.

CONCLUSION

This paper was originally prepared in 1990 and has been revised to reflect some recent changes in calculator use in Japan. Japanese education is experiencing a period of rapid transition, and support is increasing for calculator use in schools. For the twenty-first century, we should construct a curriculum that uses calculators. In this curriculum, "mathematics for all" should be emphasized as a goal, and problem solving should be integrated into all content. In this curriculum the calculator would be used more freely. In conclusion, figure 11.3 is proposed as a framework to guide calculator use in mathematics education in the future.

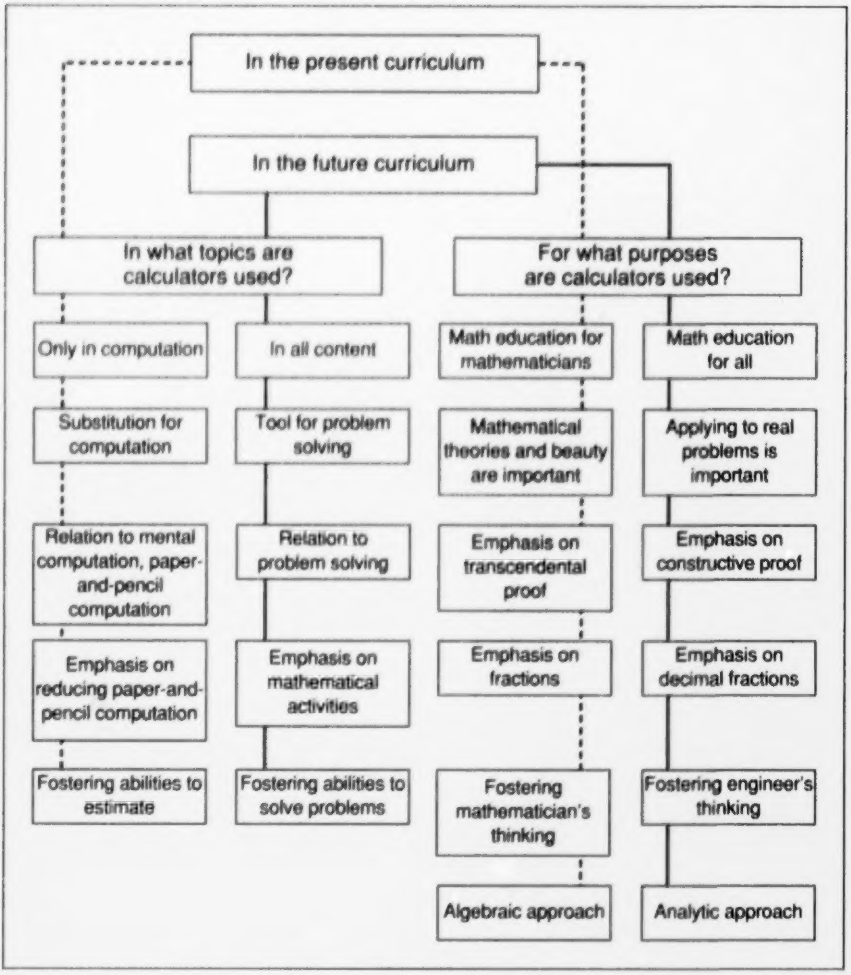

In the present curriculum
In the future curriculum
In what topics are calculators used?
For what purposes are calculators used?
Only in computation
In all content
Math education for mathematicians
Math education for all
Substitution for computation
Tool for problem solving
Mathematical theories and beauty are important
Applying to real problems is important
Relation to mental computation, paper-and-pencil computation
Relation to problem solving
Emphasis on transcendental proof
Emphasis on constructive proof
Emphasis on reducing paper-and-pencil computation
Emphasis on mathematical activities
Emphasis on fractions
Emphasis on decimal fractions
Fostering abilities to estimate
Fostering abilities to solve problems
Fostering mathematician's thinking
Fostering engineer's thinking
Algebraic approach
Analytic approach

Fig. 11.3. Place of calculator use

12

Cognitive and Metacognitive Processes in Mental Computation and Computational Estimation

Judith T. Sowder

THE following problem was given to college students of varying mathematical backgrounds (Levine 1982) and to professional mathematicians (Dowker 1992):

Estimate the product of 76×89.

College students rounded either to 80×90 or to 100×100, or attempted to mentally carry out the paper-and-pencil analog. Mathematicians, however, used a variety of unique strategies to find the estimate, including $75 \times 88 = 0.75 \times 100 \times 88 = 6600$ and $76 \times 9/10 \times 100 = 7600 \times 9/10 = 6800$.

The successful strategies used in this problem vary in level of sophistication, but in each, estimation can be seen to have two components, number approximation and mental computation. Each component is an important process and depends on the understanding and coordination of basic arithmetic skills and concepts. The following sections explore these two component processes. The metacognitive processes involved in computational estimation and mental computation are next discussed, together with the relationship of these two processes to number sense. Finally, implications for instruction are addressed.

PROCESSES USED IN COMPUTATIONAL ESTIMATION

Number Approximation

Number approximation refers to the selection of an approximate value to replace or represent a particular number. This selection process usually involves rounding, but rounding in a far more flexible manner than is taught in most U. S. classrooms. Proficiency in flexible rounding requires that the individual have an intuitive notion of the magnitude of the number in question. This intuition is the product of practice in comparing and ordering numbers—practice based on understanding. An individual who has acquired this proficiency will frequently use "benchmarks" when rounding, particularly when rounding with fractions. For example, the fraction 4/17 would be recognized as close to 4/16, or 1/4, a fraction much easier to use in mental computations than 4/17. Another hallmark of flexible rounding is the use of "nice numbers," that is, the approximate numbers are selected because they lend themselves to easy computation. For example, in selecting approximate numbers to estimate 365 ÷ 67, the approximations 350 and 70 would be selected in preference to 370 and 70 because 350 is a multiple of 70.

The ability to round to appropriate approximate numbers depends on a good understanding of place value (for whole and decimal numbers) and of the conventions involved in symbol use (particularly for fractions), as well as a basic knowledge of arithmetic facts to guide choices of nice numbers. Yet there is considerable evidence that the rounding procedures taught to children in American schools actually hinder their development of flexible rounding skills. Sowder and Wheeler (1989) found, for example, that fifth- and sixth-grade students objected to rounding 267 to 250 rather than to 300 even when it was more appropriate to do so.

Number approximation is intimately related to concepts of number magnitude, that is, knowing what a number is close to and ordering two or more numbers by size. Although most children develop (often seemingly on their own) an algorithm for comparing whole numbers (Sowder and Wheeler 1987), many never develop the ability to compare and order rational numbers. For example, children often claim that 3.53 is larger than 3.7 because 53 is larger than 7 (Sackur-Grisvard and Leonard 1985). Similarly, they claim that 1/3 is smaller than 1/4 because 3 is smaller than 4 (Behr et al. 1984) and that 2/3 and 3/4 are equal because each has "the same number of pieces left over" (Peck and Jencks 1981, p. 344). Their difficulties with solving tasks that demand an understanding of rational number size was especially evident in a study by Threadgill-Sowder (1984) in which, for example, only 19 percent of students in grades 6–9 used 0.5 or 1/2 or 50% to replace 0.52 when estimating 789 × 0.52 (the others used 0 or 1). Also, when asked to estimate the sum of 9/31 and 4/9, none recognized that the benchmark 1/3 (or 1/2 for 4/9) could be used to estimate the sum of the two fractions.

There is very little instructional time devoted to developing concepts of number magnitude. Yet, in a study by Sowder and Markovits (1989), instructional units of approximately ten days on fraction size and ten days on decimal number size at the fourth-grade level were found to be quite effective in developing a good sense of rational number size.

Mental Computation

Two basic processes work in tandem during mental computation (Sowder 1988). The first process is the reformulation of the problem into one that is dependent on a knowledge of basic arithmetic facts and skill in working with powers and multiples of 10. The choice of reformulation is related to the second process, which is the application of properties of arithmetic operations to obtain a solution. For example, 43 – 28 might be mentally reformulated as 45 – 30, and the solution thought of as using the number facts (here used with multiples of 10) 40 – 30 and 10 + 5. Others might think of this problem as (30 + 13) – (20 + 8). In this instance, the computations that follow the applications of the associative and commutative properties of addition to this expression depend on knowing the basic facts 30 – 20 and 13 – 8. Using another example, one individual might reformulate 5 × 48 as 5 × (40 + 8), so that after applying the distributive property, the resulting computations of 5 × 40, 5 × 8, and 200 + 40 depend only on the knowledge of basic multiplication and addition facts, together with place-value understanding applied to powers and multiples of 10. Another might reformulate the problem as 10/2 × 48, resulting in (10 × 48) ÷ 2, where 480 ÷ 2 is easily solved with extended basic fact knowledge. The interrelatedness of the two processes is critical; the reformulation is selected *because* the individual knows what properties can then be used. An individual who does not know that $a - b = (a + c) - (b + c)$ would not think to reformulate 43 – 28 as 45 – 30. (Of course, the strategies illustrated here are only a few of the many possible for each problem.)

In comparing successful and unsuccessful mental calculators, Hope and Sherrill (1987) found on the one hand that skilled mental calculators used a variety of strategies notable for their efficiency and reduced memory demands. One technique they used was to frequently work from left to right in order to avoid the regrouping demanded by the right-to-left algorithms with their concomitant need to recall interim calculations. Unsuccessful individuals, on the other hand, ignored even the most obvious number properties and favored the mental analog of the pencil-and-paper algorithm.

Studies of calculations by unschooled individuals also offer insight into the relation between skill in mental computation and an understanding of the structure and properties of the whole-number system. Unschooled children of the Ivory Coast Dioula people who worked with their parents in the marketplace became very skilled at mental computation, using a repertoire of number facts together with a regrouping strategy based on the decomposition of numbers, associativity, and commutativity. The mental computation procedures of their schooled Dioula counterparts were dependent on the standard algorithm (Ginsburg, Posner, and Russell 1981). Similarly, Brazilian street children given mathematical problems embedded in real-life contexts were quite capable of performing the mental computation necessary to solve the problems (Carraher, Carraher, and Schliemann 1985). However, they were less successful on similar problems given in a more formal context where they attempted to use school-learned procedures.

BEYOND COGNITIVE PROCESSES

Self-monitoring and Regulating Performance

The analysis of computational estimation given above shows it to be the composition and integration (coordination) of two processes, number approximation and mental computation. These processes in turn depend (1) on the ability to compare numbers and coordinate one's knowledge of basic facts, powers and multiples of 10, place value, symbol systems, and properties of operations and (2) on the understanding of the role of approximate numbers and how different choices of approximate numbers can lead to different but acceptable outcomes. Mental computation was shown to be the result of coordinating two closely related processes: reformulating a problem into one where a knowledge of basic facts can be used, and applying arithmetic properties to the reformulated problem. Yet these analyses are still incomplete because they do not account for the differences in strategy selection by novices and experts noted at the beginning of this paper. Individuals who are considered to be successful at mental computation and computational estimation are usually characterized as flexible, self-confident, tolerant of error in estimates, and disposed toward making sense of the mathematics being undertaken and seeking reasonableness in results. The following protocols exemplify these characteristics.

1. A seventh grader's estimate of 1 7/8 × 1.19 × 4:
   Round to 4 × 1.2 × 2, which gives 4.8 × 2, or 9.6, but this is about 1/8 too high because of rounding 1 7/8 up to 2. So knock off 1/8 of 9.6, or 1.2, which gives 8.4. (Reys et al. 1980, p. 173)
2. *The work of a highly skilled thirteen-year-old mental calculator, given the problem 123 × 456:*
   I first thought that 456 × 123 is 152 × 41 × 9. Then I thought 19 × 41 × 8 × 9. I did 779 [19 × 41] × 8 by thinking 5,600, 560, and so 6,160; 6,160 and 72 is 6,232. Finally, I multiplied 6,232 by 9 thinking 9 × 62 is 558, which I know, and so 55,800 and 9 × 32. Finally 55,800 and 288 is 56,088. (Hope 1987, p. 335)
3. *A mathematician is asked to estimate 25,410 ÷ 65:*
   25,410 ÷ 65 is like dividing 25,410 ÷ 67, which is like dividing by 2/3, which is multiplying by 3/2 or 1 1/2. 26,000 × 1 1/2 is 39,000. Adjust the decimal point, and we get 390. (Dowker 1990, p. 8)

4. *A prospective elementary teacher, asked if her success at estimation was due to effort:* In fact, to me, it's not effort at all. I don't have to go, "Oh, gosh, I'm going to estimate." It just flashes in my head, and I don't even think about it. It's easy for me, and it's not a struggle. (Sowder 1989, p. 187)

These protocols take us from the realm of cognitive processes to that of metacognitive processes, beliefs, and affect. The individuals in the first three protocols exemplify the self-monitoring and self-regulation of their decisions and actions during the computational process. It is these metaprocesses, perhaps more than any others, that distinguish those who are successful at mental computation and computational estimation from those who are not. Flexibility is but a manifestation of self-monitoring.

The knowledge that one has control over the entire computational process is manifested in the self-confidence of the student in the fourth protocol. Self-confidence makes it possible to believe that the process makes sense and that the result is reasonable. This disposition toward sense-making is critical. Resnick (1986) has hypothesized that the major difference between those who succeed in mathematics and those who don't is that good mathematics learners expect mathematics to make sense.

Acquiring Expertise in Alternative Computational Methods

How is it, then, that some individuals develop this expertise and others do not? The issue of schooling is certainly one part of the answer and will be considered in a later section, but first it is worthwhile to examine some current thinking on the matter of expertise.

Hatano (1988) distinguishes between two types of expertise—routine and adaptive. A person with routine expertise can solve routine problems quickly and accurately using automatized procedures. Such expertise is acquired when one repeatedly solves problems of the same type, for which efficiency and right answers are valued but understanding is not. Hatano's example of routine expertise is the expertise required for mental abacus operation.

In the United States, the acquisition of routine expertise in carrying out paper-and-pencil calculations has long been an objective of schooling in mathematics, and certainly before the advent of calculators this objective was a necessary one. There will probably always be skills we wish to make routine. However, it is questionable whether mental computation and computational estimation can ever be completely routinized. Adaptive expertise, requiring an understanding of how and what procedures work and how these procedures can be modified to fit the constraints of a problem, seems more appropriate for the alternative calculation methods, namely, mental computation and computational estimation. Hatano has suggested that three conditions are necessary for the acquisition of adaptive expertise: (1) encountering novel types of problems continually, (2) being encouraged to seek comprehension, and (3) being free from pressure to obtain external reinforcement. To encourage adaptive expertise with computational estimation, then, students should see a variety of estimation problems requiring them to find or "invent" appropriate algorithms using their understanding of the problems and the appropriate degree of accuracy needed for each. The emphasis should be on whether the answers make sense rather than on whether they are right or wrong.

Studies of expertise in cognitive domains have also led Dreyfus and Dreyfus (1986) to hypothesize that practicing rules does not necessarily develop true expertise, which in fact includes many capabilities that are acquired independently of rules. Greeno (1991), in reflecting on the Dreyfus and Dreyfus model of expertise, noted that the "nondeliberate aspects of skill involve recognition of holistic patterns and interactive engagement with situations including flexible adjustment of nuances and configurations that the person does not consider consciously, and probably never considered consciously" (p. 208). An expert chess player, for example, who depends more on holistic board configurations than on rules, shows such expertise. The expert mental calculator mentioned above and discussed in more detail in Hope's (1987) report also appeared to deal with problems holistically rather than in terms of rules. "Deliberate memorization of mental calculation procedures and numerical equivalents did not seem to figure prominently in the development of Charlene's abilities" (p. 339). Charlene's self-monitoring seemed to include a constant check for patterns and for known facts, such as squares and primes. When Hope commented that her knowledge of squares might be considered memorizing, she exclaimed, "But that's not memorizing. That's knowing and thinking" (p. 339).

Number Sense as Expertise in the Domain of Number

One of the ways we could characterize the expertise demonstrated in the four earlier protocols is to say that these individuals possess number sense. Number sense enabled them to deal with their problems holistically, to "know" what to do without the need to articulate a procedure before even beginning, to monitor and regulate the process of solving their problems. This insight into number problems is akin to problem solving in other areas of mathematics. Both represent a certain way of thinking. Neither can be easily defined. Yet a lack of a standard definition for problem solving has not stopped researchers from studying this phenomenon nor curriculum developers from planning for it. Just as we recognize expertise in problem solving by the creative and nonroutine approaches taken, so we recognize expertise in nonstandard computation by the creative and nonroutine solution paths taken.

Number sense can be characterized as a well-organized conceptual network that enables a person to relate number properties with operation properties. It can be recognized by the ability to compose and decompose numbers and move flexibly among different representations, to compare and order numbers, to use benchmarks, to deal with absolute magnitude of numbers, to link numeration, operation, and relation symbols in meaningful ways, to mentally calculate and estimate using "invented" strategies, to understand the effects of operations on numbers, and to be disposed to make sense of numbers (Sowder 1992). These characteristics, which individually tell us little about number sense, collectively tell us a great deal about one's ability to work with alternative computational methods. Conversely, many believe that instruction and practice with mental computation and computational estimation have the power to increase number sense when students are encouraged to seek out alternative strategies and reflect on them.

INSTRUCTIONAL CONSIDERATIONS

Role of Intuition

We know that children acquire many mathematical understandings that are not directly learned in a classroom setting. Studies of children's knowledge of mathematics when they enter school (Carpenter and Moser 1984) and of mathematics learned in out-of-school settings (described earlier in this paper) show us that children bring with them into the classroom a great deal of intuitive knowledge about mathematics. The out-of-school studies also show that when children with strong intuitive knowledge about mathematics received formal training in mathematics, they seldom made any connections between marketplace mathematics and school mathematics. There is other evidence that school arithmetic is frequently self-contained and separate from everyday mathematics (Cobb 1988; Resnick and Omanson 1987). One reason that children fail to make the connections between concepts and symbols is that they tend to separate quantitative and symbolic representations of number, and consequently the syntax and semantics of formal mathematics become detached (Resnick 1987). Also, the mental computation and estimation procedures acquired outside of school are often context-bound and not generalized to other situations. One of the greatest challenges of instruction is to build connections between intuitive knowledge and formal symbols and rules so that students' knowledge can be generalized to many contexts and situations.

A Classroom Climate to Promote Number Sense

Greeno (1991) has used a spatial metaphor to characterize the environment in which number sense can develop. In this metaphor, knowing the domain includes "knowing what resources are available in the environment as well as being able to find and use these resources for understanding and reasoning.... Learning the domain...is analogous to learning to live in an environment.... and learning how to use the resources there in conducting your activities productively and enjoyably" (p. 175). When this metaphor is applied to number sense, we can think of numbers and quantities as objects in the domain and relations and operations as a structure for the domain. "People with number sense know where they are in the environment, which things are nearby, which things are easy to reach from where they are, and how routes can be combined flexibly to reach other places efficiently" (p. 185). One aspect of this metaphor is that a guide is very useful in learning one's way around in a new environment. The teacher, as guide, must be very familiar with the environment in order to lead students to feel comfortable in this space.

There is some controversy about how to teach mental computation and estimation. These topics are frequently taught as specific procedures, but many people are beginning to question whether we might be more successful posing situations and then guiding students through the types of patterns and strategies that emerge. Such an approach is more in keeping with the notion of numbers and quantities as an environment in which to move, work, and play. "For me, this is a different way to view mental computation and estimation, and is a significant departure from conventional ways of thinking about these topics. It is number sense that should drive these topics rather than formal properties of the real number system under various operations" (Trafton 1989, p. 76).

If we consider the metacognitive processes of self-monitoring and self-regulating to be as essential as the cognitive processes associated with the development of alternative computational procedures, then we also need to consider nonconventional ways of approaching instruction. This is the challenge of teaching for number sense.

13

Treating Estimation and Mental Computation as Situated Mathematical Processes

Edward A. Silver

This paper explores some potentially fruitful theoretical formulations for research on alternatives to paper-and-pencil computation, namely, estimation and mental computation. This author approaches the topics of estimation and mental computation as an "outsider" who has conducted research primarily in the area of mathematical problem solving rather than computation and is interested in estimation and mental computation as these mathematical processes are used in *problem situations*. The central thesis is that research on these topics will benefit from more focused attention on the *situations* in which mental computation and estimation are used.

The paper begins with a brief discussion of cognitive theory, with special attention given to the emerging notion of situated cognition. This particular theoretical position will be explored because it has not yet become established in mathematics education research and because it may provide support for people seeking to make progress investigating contextualized aspects of estimation and mental arithmetic. Next, some problem-solving research that deals with situational factors as influences on children's problem-solving performance will be examined. Finally, a possible direction for research that might emerge from a serious consideration of mental computation and estimation as situated mathematical processes will be discussed.

KNOWING IN CONTEXT: A CHALLENGE TO COGNITIVE THEORY

There are many theoretical perspectives that might form a suitable basis for instructional research on computational alternatives. Case (1989) has reviewed several of these and suggested productive directions, starting from the perspective of the empiricist, the logico-mathematical developmentalist, and the cognitive structuralist. Most of the work done to date on the topics of estimation and mental computation has proceeded from these theoretical assumptions. For example, studies of expert estimators or mental calculators have attempted to identify and describe the cognitive processes and mental structures that appear to account for the expert performance (e.g., Nohda et al. 1990; Reys et al. 1982). Other studies have examined children's development of competence on tasks of increasing complexity as they mature (e.g., Case and Sowder 1990; Sowder and Wheeler 1989). These lines of work have been solidly grounded in, or closely allied with, cognitive theory, and they represent fruitful directions for further research.

Most of the research has been influenced by a general information-processing framework. According to this view, knowledge and cognitive processes are dissected into components and are hypothesized to reside in cognitive structures in the mind of the knower. As Greeno has described it:

> An information-processing analysis assumes that a person constructs a representation of the situation and his or her goal and reasons by manipulating the symbols in the representation. The person's knowledge includes information structures corresponding to the concepts and propositions.... Analysis of a specific cognitive capability, in this framework, involves hypothesizing a set of processes for representing situations, a set of knowledge structures—most frequently in the form of schemata—that provide organization for the information on the reasoning processes, and a set of processes that make inferences of various kinds. (1989c, p. 49)

This general information-processing framework has been used productively to analyze many areas of mathematics learning and performance, including estimation and mental arithmetic. Nevertheless, there are important aspects of mathematics knowledge and its use that are not completely captured within this perspective. An alternative theoretical perspective may be helpful.

Some recent cognitive research has begun by defining learning and expertise in task domains not from the perspective of knowledge structures or abstract general mental processes but rather from the perspective of the use of knowledge in practice (Brown, Collins, and Duguid 1989; Greeno 1989a; Pea and Greeno 1990). The word *practice* is used here in the sense of Scribner (1984), who used the term to describe a culturally organized set of significant activities with its own technology and symbol system. The notion that expertise is contextual has entered the theories of some cognitive scientists from at least three directions: social theories about language and the development of thought, anthropology and the study of situated cognition, and the history and philosophy of scientific domains.

Little will be said about the social theory roots; the interested reader can find an excellent account in Cole and Griffin (1980). Anthropologists have found that a contextual view of expertise is useful as they have tried to describe the nonformal knowledge of unschooled but highly skilled practitioners in various complex performance domains, such as navigation (Hutchins in press) and tailoring (Lave 1977). Other support for the view that context plays an important role in understanding expertise comes from studies that have emphasized the study of the acquisition and use of mathematical knowledge from the perspective of particular cultural practices (e.g., Carraher, Carraher, and Schliemann 1985; Saxe 1988) or related work that has questioned the value of knowledge and principles acquired in academic settings for the purposes of practical problem solving (e.g., Lave 1988; Rogoff and Lave 1984). In general, these studies have pointed not only to the situated nature of much quantitative reasoning in out-of-school settings but also to the importance of situations in evoking particular kinds of performances. In general, this work, along with several decades' work in psychology and education that has noted a general lack of a transfer of knowledge across tasks and situations, has caused many to question a theoretical perspective that posits abstract knowledge in the head of a knower moving freely among situations to which that knowledge might be applied. This has led to the development of an alternative perspective that places performance contexts more centrally at the theoretical core; namely, situated cognition (Brown, Collins, and Duguid 1989). (Some researchers in this field have explored what they call *oral mathematics*, a term that refers to non–paper-and-pencil mathematics. This oral mathematics, which is apparently used extensively outside the classroom in everyday quantitative situations, can be thought of as *situated mental computation*, often involving approximate numbers. It is interesting to contrast this notion of oral mathematics with Pestalozzi and Colburn's use of the same term to denote classroom exercises in rote, mental arithmetic [cf. Reys and Barger, chap. 4 in this volume].)

A third impetus for the notion of highly contextualized expertise has been developments in the philosophy of science that have suggested that understanding science requires understanding the practice in which scientists engage. In the area of mathematics, the work of Lakatos (1976) on the social processes of debate and argumentation in the development of mathematical ideas, the work of Toulmin (1958) on the development of disciplinary concepts through a process of selection and debate, and the work of Kitcher (1983) on the importance of shared meaning as well as shared terminology and the importance of metamathematical aspects of new developments in mathematics have been particularly influential in building a foundation for this new epistemological view, which emphasizes notions of disciplinary practice, social construction of knowledge, and situated cognition.

Situated cognition is a theoretical position designed to explain the relationship between knowing and doing. The theory of situated cognition suggests, as its name implies, that all knowledge is situated in contexts; that is, knowledge resides in a jointly constructed space of mind and of the situation in which the mind finds itself confronted with a problem (Greeno 1989a, 1989b). Although somewhat controversial and not yet widely accepted in the research community, this view may provide the needed theoretical basis for analyzing some important aspects of mathematical cognition.

As Resnick (1989) has noted, work done from the general information-processing perspective has tended to incorporate two fundamental assumptions, which she calls decomposition and decontextualization:

> The decomposition assumption refers to the notion that competence can be completely defined by a collection of independent elements of knowledge or skill.... The decontextualization assumption refers to the idea that competence exists independently of the performance that it enables; that there is some pure or abstract form of knowing that remains intact no matter what the conditions of its use; that knowledge is fully defined as something inside an individual's head, independent of the situation in which the individual acts. (P. 35)

These assumptions are fundamental impediments to efforts to understand such widely observed phenomena as students who possess certain competencies but fail to demonstrate them on specific tasks that call for the competencies or students who appear to lack a competence in classroom situations but perform well on apparently similar tasks in out-of-class settings. Although one can study mental arithmetic and estimation as decontextualized and decomposed processes, there may be much that is gained by setting aside these assumptions and treating them instead as situated processes. In fact, I believe that estimation and mental computation are especially well suited to a more contextualized perspective.

Consider, for example, the oft-noted tendency of students to resist using estimation strategies to answer estimation problems. The most commonly observed behavior is for children to calculate an exact answer and then round off the answer to produce a number that appears to be more like an estimate. Although somewhat difficult to understand and interpret from the perspective of mental structures, this behavior may be better understood from a more contextualized perspective. In particular, it can be argued that children have a general tendency to seek exact answers to computations, especially when presented in school or school-like settings. In the context of school mathematics, this general tendency would be honored and supported by typical classroom teaching practice, in which exercises and problems that require exact answers are posed. Given the general tendency of students to seek exact answers, given the general similarity between the stimuli in tasks requiring exact answers and estimates (e.g., same symbol systems, both presented as stories), and given that the two kinds of tasks are both likely to be seen by students as school tasks, the behavior of the students is not surprising. They are only slightly modifying the performance of exact computation that is generally valued and rewarded in the school situation.

The situated perspective may not only help understand certain students' errors and failures but may also assist in designing research and instruction to develop a more solid grounding in the processes of estimation and mental computation. In order to do so, we will need to examine the situations or settings in which these processes are used. Several authors (e.g., Hope 1986; Trafton 1978, 1986; Usiskin 1986) have provided a *use-oriented* rationale for the importance of estimation and mental computation as topics in the school curriculum. With respect to estimation, for example, it can be argued that only a few situational characteristics give rise naturally to the use of estimation. In particular, estimation is natural or useful in these three settings: (a) prior to engaging in an exact computation in order to gain advance information that might aid in error detection or the selection of a computation method; (b) after an exact computation in order to check the reasonableness of the obtained answer; or (c) as an alternative to an exact computation in settings for which the numbers are infelicitous in some way, such as being very large or very small, or being unknown or impossible to know exactly. These three estimation situations could form the basis for estimation instruction, in which the use of estimation is modeled and practiced in these three settings and particular estimation strategies emerge in the situated use of estimation as a mathematical process. Such instruction would likely look quite different from current instruction, which tends to teach in sequence a set of decontextualized estimation strategies that have been arranged into a hierarchy of hypothesized complexity. Such instruction would follow naturally from an attempt to implement many of the curricular and instructional changes suggested by the NCTM *Curriculum and Evaluation Standards* (1989), such as integrating richer, more situational problems into mathematics instruction or placing greater emphasis on the interpretation, rather than simply the computation, of problem solutions.

MATHEMATICS AND SENSE MAKING

This section reviews some of my work on problem solving that has also dealt with the importance of context, especially with respect to the interpretation of problem solutions. It will concentrate on work that may have particular relevance to the topics of mental arithmetic and estimation.

In a brief but persuasive article, Carraher (1989) argues that the time may be right for a fruitful cross-fertilization of paradigms in research on children's thinking and use of mathematical knowledge. In commenting on several analyses (e.g., Baranes, Perry, and Stigler 1989; Carpenter et al. 1980; Carraher, Carraher, and Schielmann 1985) of children's ability to solve computation tasks, school-like story problems, and contextualized problem tasks, she demonstrates that different theoretical assumptions can influence both the nature of the data considered and the interpretation of results: "When researchers start from different conceptions, they ask different questions about their data and find different answers. It seems, however, that the time for rapprochement has come" (Carraher 1989, p. 322). Research on the interpretation of solutions to story problems is offered as an example of how the information-processing perspective can be integrated with a situated cognition perspective to provide a more complete understanding of children's performance.

There are many anecdotal reports, and some empirical confirmation, that children do not generally develop in school a disposition toward making sense of numbers or, more generally, of any mathematics they learn. Data from the Fourth National Assessment of Educational Progress (e.g., Kouba et al. 1988; Brown et al. 1988) indicate that many students in the United States see mathematics as a subject that is not creative, that consists mainly of facts to be memorized, and that is about symbols rather than about ideas. The prevalence of such views among school mathematics students provides the underlying support for a belief that mathematics is not necessarily supposed to make sense and that learning mathematics has little, if anything, to do with sense-making.

Division with Remainders

The assertion that students tend to see mathematics as disconnected from sense-making is confirmed in solving story problems involving division with remainders. For example, "The science teacher at Marie Curie School has been given 730 frogs. The frogs will be kept in tanks. Each tank holds 50 frogs. How many tanks are needed to hold all the frogs?"

The widespread failure of American students to succeed in solving problems involving whole number division and remainders has been documented through the National Assessment of Educational Progress and several state assessments. Only 24 percent of a national sample of thirteen-year-olds was able to solve correctly the following problem, which appeared on the mathematics portion of the Third National Assessment of Educational Progress (NAEP 1983): "An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?" A similar division problem appeared on the 1983 version of the California Assessment Program (CAP) Mathematics Test for Grade 6 and was answered correctly by only about 35 percent of the sixth graders in California. In both assessments, students commonly erred by choosing non–whole-number answers.

To understand better the basis for the observed difficulty that students have in solving division problems involving remainders, several investigations have been conducted with students in grades 6, 7, and 8 (e.g., Silver 1986, 1988a; Silver, Mukhopadhyay, and Gabriele 1992; Silver, Shapiro, and Deutsch 1993; Silver and Smith 1991). Overall, the findings of these investigations suggest that students appear to have little difficulty recognizing that division is an appropriate mathematical operation to be used in solving such problems and that they appear to have little difficulty carrying out the required computation. The failure to solve such problems successfully in school contexts appears to be directly related to students' failure to "make sense" of their computational result with respect to the problem situation.

Although students generally have little difficulty with different forms of expression for remainders in routine computation settings (e.g., 12 R6, 12 1/2, or 12.5), they often experience considerable difficulty when the computation is embedded in a problem situation. One source of the difficulty is that the same symbolic expression of a division problem can represent very different problem situations and have different answers that depend on important aspects of the situational context and the quantities involved in the problem. For example, consider the following problems: "Mary has 100 brownies, which she will put into containers that hold exactly 40 brownies each. (1) How many containers can she fill? (2) How many containers will she use for all the brownies? (3) After she fills as many containers as she can, how many brownies will be left over?" To solve each of these problems, one would perform the same calculation, 100 ÷ 40, but give a different answer to each problem. In the first problem, a quotient-only problem, the remainder is essentially ignored and only the quotient is given as the answer to the problem. For the second problem, an augmented-quotient problem, the existence of a remainder leads us to increment the quotient when answering the question. In the third problem, a remainder-only problem, the correct answer is the remainder itself.

Unlike most story problems that students solve in elementary school, sense-making is not optional activity for these problems. Since the same computational result is obtained for each problem, these problems do not allow successful solution without semantic processing; that is, without making sense of the situation, the quantities involved in the problem, and the context. A successful solution appears to depend to a great extent on mappings between and among at least three referential systems: the story text, the story situation, and a mathematical model. The distinction between the story text and the story situation, which has been articulated by Kintsch (1986), appears to be of critical importance in the solution of division story problems involving remainders, and it may help inform research on estimation story problems as well.

Consider a hypothesized version of a successful solution of the brownie problems from this perspective. A successful solver would map from the story (natural language) text a representation of the problems into a mathematical model representation of 100 ÷ 40, then perform the indicated computation within the referential system of mathematics, expressing the resulting answer with an appropriate mathematical representation. The solver would then map the computational result back either to the story text representation or to the implied story situation (in the "real world") representation in order to decide how to treat the quotient and remainder. Through such a process, the successful solver would finally obtain suitable mathematical and natural language representations of the solution that have accompanying interpretations and validity within the referential systems of real-world situations and the knowledge domain of mathematics. Figure 13.1 gives a schematic representation of the mappings involved in this idealized problem solution.

In an early study (Silver 1986), it was hypothesized that students were failing to attend to relevant information implicitly represented in the problem situation but not explicitly stated in the story text (e. g., no one is to be left behind; on some bus there may be some empty seats). Several problem variants were created that made relevant structural information more salient, and students' performance on these variants was examined. The results of that study suggested that variations in the presentation of the problem, designed to make explicit certain implicit information in the problem or the required solution, significantly enhanced students' performance. Unlike most considerations of "relevant information" for problem solving in elementary school mathematics, the focus of attention in the study was not so much on information that would enhance the mapping between the story text and the mathematical model but instead enhance the mappings between and among these two reference spaces and the story situation.

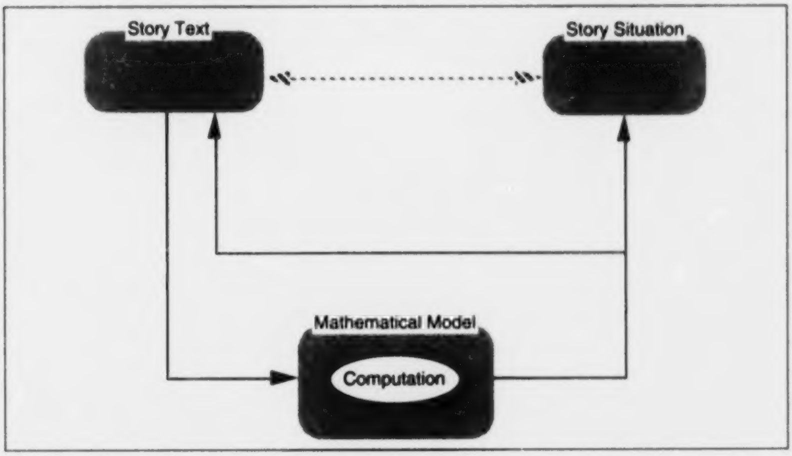


Fig. 13.1. Schematic representation of an idealized successful solution

In follow-up research (Silver 1988a; Silver, Mukhopadhyay, and Gabriele 1992), students' performance on augmented-quotient division problems and other division problem types (e.g., remainder-only problems and quotient-only problems) was examined. These studies examined the effects on students' performance of their solving other division problems that required the same computation and similar referential mappings. The results indicated that students' performance on each type of problem was enhanced by having them also solve related division problems. In general, the results were consistent with the explanation that performance was enhanced because of students' increased sensitivity and attention to the relevant semantic and referential mappings involved in the solution of the target problem. In particular, experience with the related problems may have drawn attention to the need for mapping into either the story text representation or the story situation representation after obtaining a solution to the target problem through the use of a mathematical model.

Taken together, these results and the state and national assessment findings suggest that students' failure to solve the division story problems may be due, at least in part, to their incomplete mapping among the relevant referential systems. In particular, students appear to map successfully from the problem text to a mathematical model (in this situation, a division computation to be performed), compute an answer within the domain of the mathematics model, but fail to return to the problem story text or to the story situation referent in order to determine the best answer to the question. Figure 13.2 presents a schematic representation of a hypothesized version of a student's unsuccessful solution attempt.

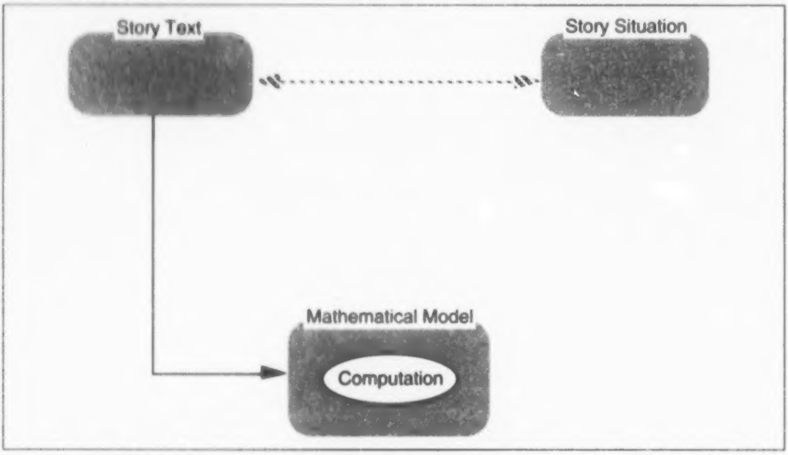


Fig. 13.2. Schematic representation of hypothesized unsuccessful solution

Although the research has focused on the interpretation of solutions with respect to situations, it should be clear that all the work described thus far has been strongly influenced by the information-processing perspective discussed earlier. In particular, the description of mental representations and mappings is deeply embedded in that framework. As the work has progressed recently, a perspective more akin to situated cognition has been found to be useful. Another study examined the problem-solving and interpretation performance of students in interview settings (Smith and Silver 1991). This study with twelve middle school students revealed that some students who would have answered incorrectly if the tasks were presented in a multiple-choice format were able to offer interesting interpretations of their numerical answers. For example, students spoke of "squishing in" the extra students or of ordering minivans rather than a full bus for the extra students. This situated thinking and reasoning remained invisible in the multiple-choice format used in the prior research.

In order to explore the general prevalence of this kind of situated reasoning, another study was recently conducted with about two hundred middle school students (Silver, Shapiro, and Deutsch 1993). In order to move away from the multiple-choice format yet accommodate the large number of students, a paper-and-pencil, free-response task was developed and administered. Much to our surprise, little evidence of situated reasoning was found in the responses of the students. In particular, nearly 60 percent of the students gave no written interpretation at all. Of the 30 percent of the students who gave the mathematically correct answer to the augmented quotient problem, about two-thirds gave an appropriate interpretation, but there was little evidence of any appropriate interpretation of alternative responses (e.g., minivans to explain fractional answers or "squishing" to explain other answers) by the remaining 70 percent of the students. After struggling with our interpretation of these findings, we discussed the results with the teachers who had administered the tasks to their students. The teachers revealed that many students discussed the problem after handing in their work and proposed alternative solutions and interpretations. This lively discussion, apparently rich in examples of situated reasoning, is evidence for the fact that students had the capability to provide interesting interpretations of their calculations. Yet, even in the free-response mode, these interpretations remained invisible to us because the students did not believe their interpretations were appropriate to be included in the work that they handed in; that is, their interpretations were not seen by them to be part of what would be accepted as mathematics work. This latter finding points to a potentially serious barrier to students' becoming adept at mathematical performances involving interpretation and sense-making.

If estimation and mental computation are seen as components of a more general construct called number sense, then there are at least two important points to be made about the findings from the research on division story problems and the general topic of number sense. One point is fairly obvious: making sensible judgments about the reasonableness of answers (e.g., recognizing that one cannot have a fraction of a bus or a fish tank) or being able to offer reasonable interpretations of non-standard answers (e.g., minivans for fractions of buses) is surely one component of what we refer to as number sense. The second aspect is somewhat more subtle and relates more to the issue of students' disposition toward sense-making. The findings regarding students' difficulties with story problems requiring division and their tendency to divorce their situated reasoning from their school mathematics solutions support the notion that school mathematics education is not currently fostering the development of a general inclination toward sense-making in quantitative matters.

A SITUATED PERSPECTIVE ON FURTHER RESEARCH

This paper has argued the value and potential power of viewing mathematical knowledge and performance from the perspective of the situations in which they are used. We have discussed situated cognition as an alternative theoretical perspective that may enhance our understanding of some important aspects of estimation and mental computation. And we have considered some research on students' interpretations of the results of their computations in a problem-solving setting from both information-processing and situated-cognition perspectives. The general theoretical perspective of situated cognition and the particular model of semantic processing discussed earlier may be useful to researchers studying estimation and mental computation as situated mathematical processes.

My final comments relate to an issue—number sense—that becomes more salient if we take seriously a situated view of estimation and mental computation.

NUMBER SENSE AND THE CULTURE OF THE MATHEMATICS CLASSROOM

As noted in many of the chapters in this volume, much of the current interest in estimation and mental computation in the United States is due to a general interest in the topic of number sense. Although it is difficult to define number sense precisely, behaviors like estimating before or after computing, judging the reasonableness of one's calculation, and using the relative size of numbers or numerical benchmarks (such as basic facts) to guide quantitative activity are all examples of sense-making actions associated with numbers and numerical activity. Thus far, instructional attempts to increase students' competence in number sense have tended to focus on discrete, teachable components of complex skills. Attempts to improve students' number sense often focus on overt cognitive behaviors whose manifestation will ensure that number sense is present and in use. For example, instructional units have been developed to help students learn many techniques for estimating the result of a numerical computation, to learn information about the precision of measurement or calculation and techniques for approximation, or to learn strategies for mental mathematics in order to increase their flexibility in calculation strategies. Although this instructional approach is likely to increase the specific cognitive competence of the students who learn from it, it may not be completely successful if it fails to embed these competencies in performances situated in authentic mathematical activity. The perspective of situated cognition may be helpful to us as we examine the design and effects of such instruction.

The situated cognition perspective also suggests that such instruction is unlikely to be completely successful in increasing students' number sense if it fails to address students' dispositions toward numerical activity, or more generally toward mathematics. It is apparent from much of the research discussed above that students do not believe that school mathematics makes sense or even that it should make sense. Unless situations are provided in which students can participate in the authentic sense-making aspects of culture in which mathematics is used, it is unlikely that they will alter that belief. Decomposed and decontextualized instructional sequences to promote mental computation and estimation are likely to leave untouched this important aspect of the culture of the mathematics classroom. To promote the development of a finely tuned number sense, what is needed is nothing short of cultural revolution in school mathematics education.

Regardless of how well the instruction is designed, isolated instruction on individual components of number sense is unlikely to lead to the development in students of an adequate sense-making disposition with respect to quantitative processes and products. Considering the division example discussed above, it is clear that teaching students simply to answer such questions correctly is not the entire story. Our curriculum and our instructional methods must be restructured to emphasize sense-making in all areas of mathematics instruction and at all grade levels from kindergarten to grade 12. Moreover, our research needs to be directed toward understanding children's sense-making capabilities at much deeper levels so that these understandings can be built on rather than ignored in classroom instruction.

As reform efforts proceed, care must be taken not to delete certain curricular topics, such as work with very large or very small numbers, that can provide important situations in which students can develop number sense. Also, we would be wise to resist the temptation to believe that extensive use of calculators will automatically develop good number sense. Calculator activities can be used in situations that might promote number sense. However, they can also be used in ways that promote an overreliance on the result found in the calculator display and that increase students' failure to make sense of the situation, quantities, and numbers involved. Recall that performance on the NAEP army-bus problem was poorer in the Third Mathematics Assessment (NAEP 1983) when students used calculators than when they used only paper and pencil!

Given the need for fundamental reform, researchers may need to be more proactive in exploring these issues in a "transformative" research agenda (Silver 1988b). Studies need to be undertaken that examine the long-term consequences of early mathematics instruction that emphasizes sense-making on the part of students and teachers. Curricular and instructional experiments related to number sense also need to be conducted with older students as they learn more complex procedures and solve more difficult problems; sense-making needs to be embedded into the fabric of mathematics courses for these older students as well. Although research must certainly also be done on specific components of number sense, it would be unwise for us to believe that the sum of the cognitive components would necessarily equal the complex whole of number sense. Studies directed at students' general dispositions toward quantitative sense-making will deepen our understanding of these components. Moreover, we need research related to students' exploration of rich situations that evoke mathematical sense-making behavior and its various components, such as estimation and mental computation. This research on estimation and mental computation should, when amplified by similar investigations in related topic areas, assist us in meeting a fundamental challenge for the next decade—explaining the nature of the relationship between richly contextualized, situated, quantitative understandings and abstract, powerful, generalized conceptions of mathematical principles and structures.

14

The Cognitive Processes and Use of Strategies of Good Japanese Estimators

Katsuhiko Shimizu
Junichi Ishida

DOES the theoretical model of computational estimation based on the research with a U.S. sample (Reys et al. 1982) also apply to the Japanese? What computational estimation strategies do Japanese students use, and how do they perform? These were the primary questions of recent research on estimation performance and strategies of Japanese fifth and eighth graders (Reys et al. 1991). In that research study, twenty-one good estimators were selected, out of a sample of 466 fifth and eighth graders, on the basis of preliminary screening data collected by administering a "slide projected" computational estimation test consisting of thirty-nine open-ended items mainly from the ACE (Assessing Computational Estimation) test (Reys et al. 1982). These twenty-one students were asked to make their estimates for the problems in individual interviews. A total of twelve estimation problems were given to the "good estimators" (selected students) in fifth grade, and the same twelve plus two additional problems were given to the "good estimators" in eighth grade.

The research revealed that estimation was very difficult for almost all the Japanese students (the fifth-grade mean score was 7.39, with standard deviation of 5.16, and the eighth-grade mean score was 11.15, with standard deviation of 5.17, on the thirty-nine-item open-ended test). This result was surprising to the Japanese mathematics educators, since topics on *gaisuu* (rounded numbers) and *gaisan* (rough calculation) are taught in fourth and fifth grades. The interviews revealed that the good estimators as a whole did show the three general cognitive processes that good estimators in the American sample employed in previous investigations in the United States: translation, transformation, and compensation. Also, the Japanese students used many of the same strategies that the American students used—straight rounding, compatible rounding, and front-end techniques. Compared with the United States sample, the Japanese sample showed a greater degree of mental computation ability, less frequent errors on order-of-magnitude problems, and less willingness to accept errors. Japanese students applied algorithmic computational procedures more often, and this tendency interfered with their estimation processes.

It can be concluded, then, that on the one hand many Japanese students have difficulty with computational estimation, even though they are taught rounded numbers and rough calculation. On the other hand, the interviewed students, as a whole, employed the same key processes and strategies that good American estimators employed.

This paper reports results from three additional computational estimation research studies of Japanese students. The first two studies give a more detailed analysis of the cognitive processes of the good fifth- and eighth-grade estimators (Reys et al. 1991) mentioned earlier. The third study was designed to investigate the longitudinal development of computational estimation in Japanese students, and so similar research was conducted with university freshmen. It is hoped that these studies will furnish a better picture of the development of computational estimation processes among Japanese students and will contribute to the understanding of the general cognitive processes used in estimation.

STUDY 1: GOOD FIFTH-GRADE ESTIMATORS

In the Japanese elementary school mathematics curriculum, *oyosono-kazu* and *gaisuu* (rounded number) and *gaisan* (rough calculation) are taught in the fourth and fifth grades. For example, in one widely adopted textbook series, one lesson is devoted to *oyosono-kazu* in the fourth-grade textbook, and one lesson is devoted to *gaisuu* and *gaisan* in the fifth-grade textbook. In the treatment of *oyosono-kazu*, students are asked to answer such problems as "To which is 26 345 nearer, 26 000 or 27 000?" *Shisya-gonyu* (straight rounding) to a certain digit is also introduced to them. In the lessons on *gaisuu* and *gaisan*, the round-then-compute method with *shisya-gonyu* for the four operations is introduced as rough calculation. Since fifth graders are taught these topics, they were appropriate subjects for investigating the effects of teaching these topics.

Subjects

Four schools participated in the screening test. These schools were selected from rural and urban districts of Tsukuba and represented a range of social and economic backgrounds. The subjects were ten fifth graders who scored in the top 5 percent of 279 fifth graders who took the screening test. Their ages ranged from ten to eleven.

Interview

The subjects were identified as "good estimators" and were scheduled for a follow-up interview. Interviews were conducted within three weeks after the screening test at the student's school during the regular school day. In each interview, students were allowed to spend all the time they needed to make their estimates and to explain how each was produced. Interviewers attempted to ascertain the strategies and processes used during estimation and tried to ask each student to explain the thought processes used while deriving estimates.

Interview Problems

Four straight estimation and seven applied estimation problems were asked of the good fifth-grade estimators. Three additional problems, two involving percent (application) and one involving multistep multiplication and division (straight computation) were asked only if they said they wished to try, since these problems included topics that fifth graders had not yet been taught. The set of all interview items is shown in table 14.1.

Table 14.1
Interview Problems: Four Straight Computational Estimation and Eight Applied Computational Estimation

1. $$\begin{array}{r} 87{,}419 \\ 92{,}765 \\ 90{,}045 \\ 81{,}974 \\ 98{,}102 \\ \hline \end{array}$$
2. $8{,}127\overline{)474{,}257}$
3. $(347 \times 6)/43$
4. $(12/13) + (7/8)$
5. 486×0.24
6. Approximately what is the area of this rectangle? (A picture of a 47-by-28 rectangle is provided.)
7. At the Japan Series (World Series) 7883 cartons of juice were sold for 90 yen (cents) each. About how much money resulted from selling the juice?
8. If about 30% of the fans at the Japan Series (World Series) bought one soda, about how many sodas were sold at the series? (Included was a sign noting attendance of 54,215.)
9. The Japan Series (World Series) netted 2,139,908 yen ($2,139,908) to be equally divided among the 12 teams. About how much does each team receive?
10. This is a grocery store receipt that has not been totaled. Estimate the total. (Included was a picture of a receipt with prices: 79, 79, 44, 130, 34, 105, 57, 29, 65, 30, 31, 429, 11, 34, 8.)
11. Estimate the new price. (Picture showing a shirt with a price tag of 2,800 yen. A sign notes a SALE of 15% off marked price.)
12. Estimate the yearly salary for this position. (A sign showing this information: 40 hours a week, 50 weeks a year, Salary: 940 yen an hour.)
13. The student missed two of these problems. Which answers are not reasonable? Included was a picture of a student's paper showing:
    1. $\frac{4}{9} + \frac{5}{10} = 1\frac{5}{90}$
    2. $\frac{5}{8} + \frac{4}{7} = \frac{8}{15}$
    3. $\frac{8}{15} + \frac{11}{20} = 1\frac{1}{12}$
14. Which set of buttons is the best buy?
    a. 10 for 40 yen b. 12 for 50 yen
    c. 16 for 75 yen d. 32 for 149 yen

Results and Discussion

Table 14.2 shows the strategies used by students to solve the problems. The abbreviations represent the types of estimation strategies or responses. A blank in a column shows that the problem was not posed to that student.

Table 14.2
Strategies Used by Good Fifth-Grade Estimators

	Student				
Problem	1	2	3	4	5
1	SR	Ave (SR)	Ave	FE	FE
2	FE	Other	SR	FE	FE
3					
4	US	Algo	US	Other	Algo
5	SR	SR	Other	SR	SR
6	SR	SR	SR	FE	SR
7	SR	SR	US	FE	SR
8		FE		Other	
9	FE	SR	US	Other	FE
10	R	Group	Ave	FE	SR
11					
12	SR	SR	SR	FE	R
13	CR	Other	Other	Other	Other
14	UB	R		Other	Ratio

	Student				
Problem	6	7	8	9	10
1	Ave	SR	Other	Ave	SR
2	FE	FE	SR	CR	SR
3					
4	US	US	Algo	US	Algo
5	SR (MC)	SR	SR	SR	SR
6	SR	SR	SR	SR	FE
7	SR	SR	SR	SR	SR
8		Other		UB	Other
9	CR	SR	SR	CR	SR
10	Ave (CR)	SR	SR	CR	SR
11					
12	SR	SR	SR	SR	SR
13	Other	Other	Other	CR	Algo
14	MC (UB)	UB (R)	MC	Ratio	MC (UB)

Ave: Averaging
Algo: Use of algorithm
FE: Front end
Group: Grouping
SR: Straight rounding
MC: Did not mentally compute exact number correctly
UB: Use of benchmark
R: Somewhat round number
CR: Compatible rounding
US: Use of special number
Ratio: Use of ratio

Use of the Front-End Strategy

Good fifth-grade estimators used the front-end strategy fifteen times while solving the problems. They used this strategy mainly in problems 1 and 2. In contrast, eighth graders used the front-end strategy only two times. Therefore, this strategy can be considered characteristic of fifth-grade estimation strategies. A typical student who used the front-end strategy is Student 4. She used the front-end strategy six times in twelve problems. In other problems, she tried to modify the algorithm she had been taught or to skip a part of the algorithm ("Use of algorithm"). Her average response time was thirty-one seconds (if the time data for problem 12 are omitted, her average is twenty-four seconds). In problem 5, she said, "0.24 can be rounded to 1 or to 0." In problem 6, she said, "Both 1,500 and 800 are acceptable as estimates." These responses reveal that she used the front-end strategy but that she lacked a sense of validity for her estimates.

The front-end strategy corresponds to *kirisute* in the Japanese mathematics curriculum. *Kirisute* (round down) is taught in fourth grade. It is intended that *kirisute* will become a part of *shisya-gonyu* together with *kiriage* (round up). But, as seen in Student 4's responses, it might become an automatic skill, carried out without number sense, when acquired separately.

Prevailing Strategies and Their Patterns

Straight rounding was observed in 42 percent of the responses (48 out of 114). Straight rounding is taught as *shisya-gonyu*, and the responses reflect the effect of teaching this topic. Fifth graders mainly use straight rounding and the front-end strategy. The percent of time one dominant strategy was used ranged from 30 percent to 59 percent among fifth graders. The percentage of one dominant strategy became higher in eighth grade.

Students 7, 8, and 10 used straight rounding seven times, whereas Student 4 used the front-end strategy six times. These students also used another taught strategy and an algorithm. It seems that they learned the strategies taught to them but were not able to develop other estimation strategies. Therefore, they modified the computational algorithm to cope with some of the estimation problems.

In table 14.2, the use of one particular strategy is not observed by Students 1, 3, 5, 6, and 9. Student 1 used straight rounding five times, but he was different from the other students because he used compatible rounding, special numbers, and benchmarks. Student 3 showed a variety of responses—straight rounding and special numbers three times each and averaging two times. Student 6 also used a variety of strategies. Student 9 used straight rounding and compatible rounding four times each. These students can be characterized by their frequent use of strategies that they were not taught.

In conclusion, we observed two types of good fifth-grade estimators. One type used taught strategies throughout the problems and did not reveal any original strategies. The other was less common and used strategies that had not been taught, in addition to the taught strategies.

STUDY 2: GOOD EIGHTH-GRADE ESTIMATORS

Study 2 presents data on the cognitive processes used by the eighth graders. Although eighth graders have learned more mathematics than fifth graders and have had more experiences with various types of numbers and operations, three years have passed since they learned the topics related to estimation. It was felt that two factors that could make a difference between good estimators in fifth grade and good estimators in eighth grade would be their cognitive development and the learning of mathematical topics not directly related to estimation.

Subjects

The procedures for screening and interviewing were identical to those reported in Study 1. The subjects were eleven eighth graders who scored in the top 5 percent of the 187 eighth graders who took the screening test. Their ages ranged from thirteen to fourteen.

Interview

The subjects defined to be "good estimators" were scheduled for a follow-up interview, just as in Study 1. The same set of interview problems was given to the good estimators. But the three problems that were optional in Study 1 were always asked of the eighth graders.

Results and Discussions

Table 14.3 shows how each student selected strategies to solve the problems. The abbreviations represent the same types of estimation strategies or responses that were used in table 14.2.

Table 14.3
Strategies Used by Good Eighth-Grade Estimators

	Student				
Problem	1	2	3	4	5
1	Ave	Ave	SR	Ave	SR
2	CR	CR	CR	SR	SR
3	CR	CR	SR	SR	SR
4	CR	CR	SR	CR	SR
5	Algo	US	US	US	US
6	CR	CR	SR	SR	MC
7	SR	SR	SR	SR	SR
8	CR	CR	SR	SR	SR
9	CR	SR	SR	SR	
10	CR	SR	SR	SR	SR
11	SR	Group	SR	SR	SR
12	Comp	Other	SR	SR	SR
13	CR	CR		MC	Algo
14	UB	UB	MC (UB)	MC	MC (UB)

	Student					
Problem	6	7	8	9	10	11
1	Ave	FE	SR	FE	Ave	Ave
2	SR	CR	FE	SR	SR	SR
3	Other	CR	SR	SR	SR	CR
4	Other	Algo	US	US		Algo
5	SR	SR	CR	CR	SR	SR
6	SR	SR	SR	SR	SR	SR
7	SR	SR	SR	CR	SR	SR
8	SR	SR	SR	CR	SR	SR
9	SR	CR	CR	CR	SR	CR
10	SR	FE	SR	Group (SR)	Group	CR
11	SR	CR	SR (UB)	CR	SR	SR
12	SR	CR	SR	SR	SR	SR
13		Other	Other	CR	Other	Other
14	UB	UB	UB	UB	UB	MC

Note: Table 14.3 uses the same symbols as table 14.2 to identify strategies.

The number of different types of responses from any one student ranged from three to seven. Also, the percentages of the use of one particular strategy by a student ranged from 28 percent to 70 percent. From these responses from good estimators, two tendencies were observed.

Quick Estimation by Straight Rounding

Students with a high frequency of use of one strategy throughout the problems used straight rounding most frequently. For example, Student 3 used straight rounding ten times on the fourteen problems. He used four different strategies, which is rather few compared to other students. He produced his estimates within five to ten seconds on some problems and had a very low average response time of about twenty-five seconds. The interviewer's impression of this student was that his mental computation was good, flexibility in working with numbers was average, adjusting or compensating was rare, and tolerance for error was comfortable.

This pattern was also found in responses from Students 4, 5, 6, and 10. They used straight rounding nine times each. Compatible rounding was used only once by Student 3 and once by Student 4. Other responses were observed primarily in problems 1, 13, and 14. Their average response time ranged from twenty-five to forty-nine seconds. According to the interviewer's impression of these students, their mental computation skill was quite good (for three out of five students), their flexibility in working with numbers was average (three out of five students), they adjusted or compensated rarely (all five students), and their tolerance for error was comfortable (all students). Also, it should be mentioned that for problems 13 and 14, which required a conceptual treatment of numbers, they tended to apply mental computation directly or to modify the taught algorithm to cope with problems.

These results lead to this description of a representative estimator: A "good estimator" uses straight rounding quickly, exhibits a tolerance for error, and possesses comparably good mental computation ability. But these "good estimators" did not use compatible numbers when they rounded, nor were they confident in their estimates.

Use of a Variety of Strategies

Students 2, 7, and 9 used the most strategies on the problems. Student 2 used compatible rounding six times. In addition to this main strategy, averaging, straight rounding, the use of special numbers, grouping, and the use of benchmarks were also observed. His average response time was about thirty seconds a problem.

Student 7 used six different strategies, including straight rounding and compatible rounding four times each. He was the slowest, with an average response time of sixty-seven seconds a problem. His flexibility in working with numbers was rated good. Student 9 also used six different strategies, with compatible rounding being used six times. Her average response time was about forty seconds a problem.

These students can be characterized by their use of compatible rounding and by the variety of their responses. Their ability with mental computation was very good, their flexibility in working with numbers was good, their tolerance for error was comfortable, but adjusting or compensating was infrequent or never used. The average response time ranged from thirty-one to sixty-seven seconds. Even though they were rated better than other good estimators in mental computation, their response time was slower than that of others. This suggests that their cognitive processes may have more subprocesses than those of others, including finding numbers that are compatible, selecting suitable strategies, and so on. They were flexible in using many strategies when working problems. The only problems on which they used the same strategy were problems 6 and 14. Therefore, whatever strategy they selected seemed to depend on what they perceived as the numerical nature of the problem.

Research studies on estimation have reported that good estimators use a variety of strategies. Similarly, good Japanese estimators can be characterized by their use of compatible rounding and a variety of other strategies in addition to their use of taught strategies.

Use of Adjustment and Compensation

The number of times fifth and eighth graders adjusted their estimates is shown in figure 14.1. As seen in this figure, the majority of students adjusted their estimates only a few times, and five eighth graders adjusted estimates once or not at all. In previous research studies, adjusting and compensating were characteristic of good estimators. This characteristic was also observed in good Japanese estimators, but the frequency was not as high.

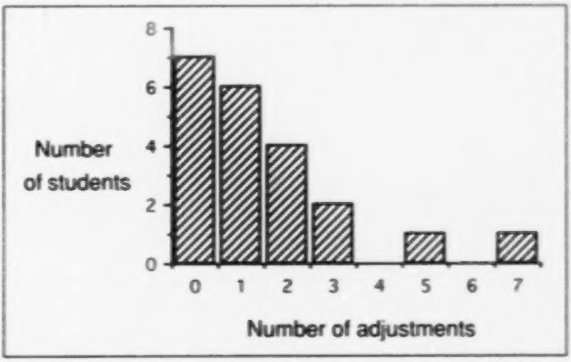


Fig. 14.1. Number of times eighth graders adjusted estimates

One student adjusted his estimates seven times. He was categorized as an estimator with flexible strategies. He adjusted estimates on problems 1, 3, 5, 6, 7, 8, and 9. This shows that he adjusted when he rounded numbers.

Another student adjusted estimates five times. His main strategy was straight rounding, which he used seven times. He could be categorized as a quick estimator who used straight rounding. He adjusted estimates in problems 1, 3, 7, 10, and 11. He seemed to adjust his estimates when there were many numbers or when the numbers were large.

These two students are different from those in the previous section. Some of those students could be categorized as flexible, whereas others could be categorized as quick. Whether the students adjusted or compensated seemed to be independent of which way they were categorized previously.

Different Strategies for Estimating

After students answered each estimation problem, the question "Can you estimate in a different way?" was asked. The number of different ways they were able to estimate is shown in figure 14.2. Fifty-five percent of the students used fewer than three strategies. Only three students used more than seven strategies for estimating. The good eighth-grade estimators found it difficult to produce an estimate a different way. For many students, it appeared difficult to accept another method than their own, since they felt there was only one way to estimate. Sowder and Wheeler (1989) reported that the acceptance by students of other estimation processes is slow to develop. This is consistent with Japanese students' difficulty in estimating different ways.

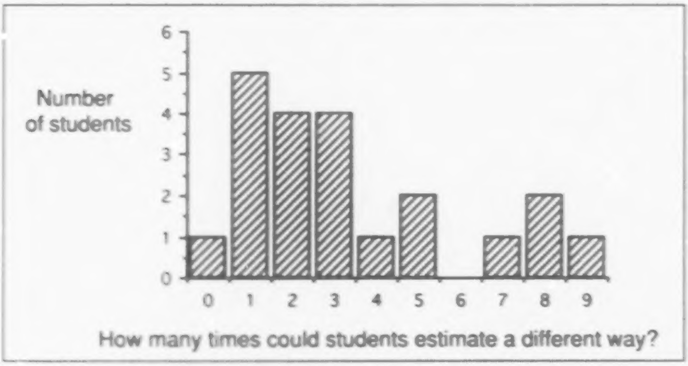


Fig. 14.2. Number of different strategies used by eighth graders

Students 1, 4, 5, and 9 used seven or more different strategies in producing their estimate. The main strategy used by Student 9 was compatible rounding. The different ways they estimated included finding other compatible numbers, using the front digit, and so on. The main strategy of Students 4 and 5 was straight rounding. They estimated in different ways, such as shifting the digit that was to be rounded. In fact, they did not use another new strategy; rather, they shifted to other digits. The impressions the interviewers had of them were that they were flexible in working with numbers, their tolerance for errors was comfortable, and they sometimes adjusted and compensated.

STUDY 3: GOOD ESTIMATORS AMONG UNIVERSITY FRESHMEN

In the Japanese mathematics curriculum, except for some choices of courses at the senior high school level, almost all students take the same courses in the national curriculum from elementary to senior high school. More than 98 percent of students go to senior high school. University freshmen can therefore be considered as a good sample of those who have finished the standard mathematics curriculum. Study 3 was conducted to investigate the cognitive processes of students in this sample.

Subjects

Sixty-nine undergraduate students in their first year at the University of Tsukuba participated in the screening test. These students were taking the course titled Theory of Educational Practice, and their majors were mainly in the humanities. Six freshmen were selected for interviewing on the basis of the test. The mean score on the test was 21.54 with a standard deviation of 5.91. The scores of the six selected freshmen ranged from 29 to 33. This sample consisted of college students but did not include all academic majors, and some of them were older than nineteen.

Interview

The follow-up interviews were conducted within one month after the screening test. The procedures for the interviews were identical with those reported in Studies 1 and 2. The set of interview problems was the same as for Study 2.

Results and Discussion

Table 14.4 shows the results on how the six university freshmen responded to the estimation problems.

Table 14.4
Strategies Used by Good University Freshmen Estimators

	Student					
Problem	1	2	3	4	5	6
1	Ave	Ave (Tr+Co)	SR	Ave (Tr)	Ave (Tr)	FE
2	SR	CR	CR	CR (Co)	SR	SR
3	CR	BC (Co)	CR	BC	SR	CR (Co)
4	US	US	US	MC (Co)	US (Co)	MC
5	CR (Co)	SR (Co)	CR	CR	CR	CR (Tr)
6	SR (Co)	SR (Co)	SR (Co)	SR (Co)	CR (Co)	SR (Co)
7	SR	SR (Co)	SR (Co)	SR (Co)	SR	SR
8	SR	SR,Ave (Tr)	CR (Tr, Co)	BC	SR	SR
9	SR	BC	FE	SR (Co)	BC	CR
10	SR	SR	SR	SR	SR	SR
11	MC,SR	BC	MC	MC	SR (Co)	Ave (Tr)
12	SR	SR (Co)	SR	MC	CTR	MC
13	UB,US	UB,US	UB,US	UB,US	UB,US	MC
14	MC	UB	MC	UB	MC	MC

New categories:
CTR: Compute then round
Tr: Translation
MC: Subject did compute exact number correctly
BC: Bold computation, which produces bold answer by computation of exact numbers
Co: Compensation

Strategies Used

Figure 14.3 shows the percentages of strategies used by students in the interview. The most commonly used strategy was straight rounding, and compatible numbers was a distant second. The front-end strategy was used very little. The average number of strategies used was slightly higher than that of the eighth graders. It was conjectured that these good estimators could select a strategy appropriate to the problem and that they knew

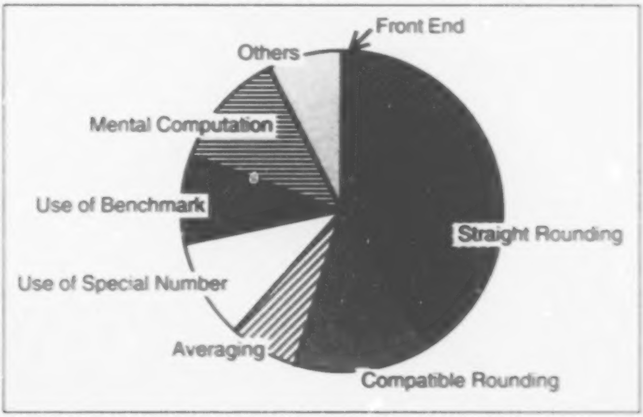


Fig. 14.3. Percent of estimation strategies used by university freshmen

enough strategies to do this. The following examples support this hypothesis: averaging was frequently used in problem 1; straight rounding in problems 6, 7, 8, and 10; compatible rounding in problems 2, 3, and 5; use of a special number in problems 4 and 13; and the use of a benchmark in problems 13 and 14. In problem 10, there seemed to be a particularly heavy reliance on straight rounding. All six students used the straight rounding strategy in this problem. They added the sixteen addends in order by applying a rounding strategy. In problem 2, which asked subjects to estimate 474,257/8127, three subjects rounded to compatible numbers such as 480 and 8 to make their estimate. In problem 5, all the subjects produced an acceptable estimate. Five students used the compatible rounding strategy. Figure 14.4 shows that as the level increases from grade 5 to college freshmen, the frequency of straight rounding decreased and compatible rounding increased. This tendency shows that these subjects were able to choose numbers that produced "nice estimates." In problem 4, four subjects used special numbers to estimate, that is, they recognized that both 12/13 and 7/8 were close to 1.

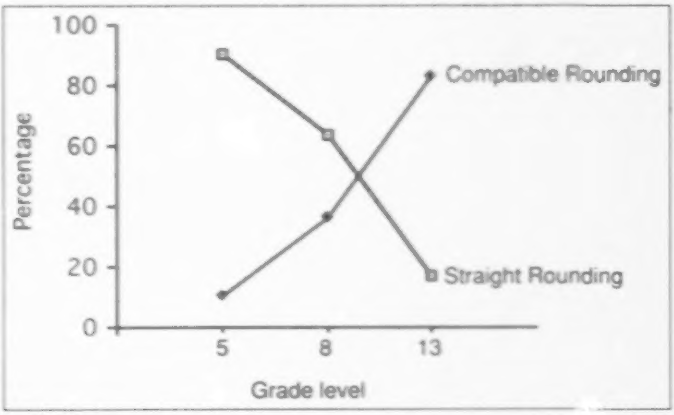


Fig. 14.4. Percent of good estimators using two different rounding strategies on problem 5 by grade level

In summary, students used a variety of estimation strategies. Furthermore, they intended to use strategies appropriate to the problems. The three principal processes (translation, transformation, and compensation) were employed in combination with one other. Because they learned estimation only in the fourth and fifth grades of the elementary school Japanese mathematics curriculum, the learned strategies they used were mainly *shisha-gonyu*, *kiriage*, and *kirisute*. But they were able to use strategies they had not been taught in combination with the three principal processes.

Three Principal Processes

The three processes were also observed to be employed in tandem. Their use could be pointed out as characteristic of good estimators within this group of freshmen. The following examples illustrate how some of these processes were used when estimating. In problem 8, Subject 2 used reformulation, translation, and compensation. Her estimation process was as follows: (1) $54{,}215 \times 0.3 = 5{,}421.5 \times 3$ (using arithmetic properties), (2) round 54,215 to 54,000 (reformulation process), (3) $5{,}400 \times 3 = 5.4 \times 3 \times 1{,}000$ (using arithmetic properties), (4) $5 \times 3 = 15$, $6 \times 3 = 18$ (averaging or transformation strategy), (5) after she averaged 15 and 18, she selected 16 (compensation). Finally, she arrived at 16,000.

Subject 3's process of estimating in problem 8 also exemplified the combined use of the three processes: He reformulated 30 percent to 1/3 and 54,215 to 54,000. He then transformed the problem structure to $54/3 = 18$, where the decomposition of a power of 10 was considered. Finally, he adjusted his estimate by subtracting 1 from 18 to get 17, and answered 17,000. Among these combined processes, the combination of reformulation and compensation was frequently observed. In problem 6, all subjects exhibited this combination of processes. Also, all subjects exhibited this combination of processes at least twice.

Flexibility of strategy and combined use of processes were found to be characteristics of undergraduate good estimators. Those two characteristics are particularly noticeable when compared with those of fifth and eighth graders. The results of this study suggest that as good estimators grow older, their estimation ability develops and becomes sophisticated, even without being formally taught any processes or strategies after fourth and fifth grade.

GENERAL DISCUSSION

Taught Strategies

The students selected for these studies developed their expertise in estimation after *gaisuu*, *kiriage*, *kirisute*, and *shisyagonyu* were taught to them in fourth and fifth grades. When comparing good fifth- and eighth-grade estimators, the few times the fifth graders used the front-end strategy and straight rounding implies that these topics are still developing, even in good estimators, at the fifth-grade level. They have learned how to round numbers and therefore might have used these strategies, especially straight rounding, on more problems than they did. It seems that they knew rounding strategies, but they still tended to rely on computational algorithms or exact mental computation. Also, in some problems, such as those including fractions, they did not use the taught rounding strategies. It seems that they could apply these strategies only on a limited number of problems.

The results show that after students have learned rounding strategies, they need to acquire efficiency with other components of estimation. The contrasting strategies used by fifth and eighth graders point to the importance of treating numbers with reference to their nature in the problem. Number sense is perhaps involved in this component of estimation.

In the university freshmen, the taught strategy "straight rounding" was commonly observed, but their reliance on this strategy could be due to a reason other than that this strategy was the most appropriate, possibly that their high mental computation ability combined with number sense made this strategy feasible. They were able to reformulate numbers, translate problems, and mentally compute quickly and accurately.

Rounding Numbers

Fifth graders do not appear to be conscious of choosing appropriate numbers when rounding. *Kiriage* is taught, for example, as "Round up the digit with which you are dealing": 534 to 540, or 625 to 700. *Shisyagonyu* is taught, for example, as "When the numeral to the right of the digit with which you are dealing is below 4, round down, and when the numeral to the right of the digit with which you are dealing is over 5, round up" (the word-by-word translation is *"Throw it, if it's below four; take it, if it's over five"*). These rules have nothing to do with what approximated number is relevant with reference to the accuracy of the estimate. Also, adjustment was observed only five times throughout the work of the selected fifth graders. These results support the view that students lack a feeling for the accuracy of estimates. The feeling for the accuracy determines what number to round to and whether adjustment or compensation is required or not.

The reformulation of the numbers in a problem includes not only the process of rounding these numbers but also the cognitive process of searching for numbers to which to round that are suitable for producing "nice" estimates. This cognitive process consists of searching sets of numbers and, at the same time, checking whether a set of numbers produces a "nice" estimate. Our eighth-grade students, who use compatible rounding frequently, were able to round appropriately, naturally, and intuitively. As shown in figure 14.4, the percentage of compatible rounding increases as they grow older. It seems to require long-term development for students to become fluent in compatible rounding.

To get "nice" estimates, students need to grasp various aspects of the nature of the numbers: how big they are, if a multiple of a certain number, whether some are close to certain fractions, how they are composed and decomposed, and so forth. The ability to grasp the various aspects of numbers is another ingredient in the cognitive process of rounding numbers. This ingredient is increasingly observed as the grade level of students increases. This ability to grasp many aspects of numbers perhaps is enhanced by long-term learning experience with various numbers.

The manner in which students see or recognize written numbers in problems is an important cognitive process in estimation. In ongoing research on estimation strategies and the performance of soroban experts (who have mastered soroban levels from 3 *kyu* to 1 *kyu*, with 1 *kyu* being the highest), one soroban teacher told us, "If they see the leftmost several digits, they will be able to produce 'nice' estimates," but the research suggests that they do not seem to do so. It should be realized that soroban experts are trained to watch and to compute each number in each place. When the number is very big, they do not have enough time to do so in an estimation screening test because they do not try to grasp the magnitude or other aspect of the number. This difference of opinion on the impact of the soroban on estimation thinking suggests that this topic needs to be carefully researched.

Mental Computation

Mental computation is another important ingredient of estimation. From the impressions of the interviewers, high ability in mental computation is like a double-edged sword. It sometimes enables students to find nice pairs of numbers, to compute numbers quickly, and to notice when estimates are unacceptable. But it also sometimes inhibits students from estimating, and they seem to be addicted to computing mentally, even when it is impossible for them to do so. When they try to mentally compute these problems, they modify or skip a part of the algorithm in order to cope with the difficulty, and they fail. This tendency is observed in fifth-grade students. In contrast with fifth graders, university freshmen seem to translate problems or reformulate numbers according to their mental computation ability.

Mental computation itself is a computational alternative and is also a component of estimation. Which type of computation is relevant depends on the problem situation and on the student's computational abilities. Students should know which process is suitable for them to use on any problem. How students decide what computational alternative is suitable for a problem will be another target of research on the cognitive processes associated with computational alternatives.

Some Characteristics of the Cognitive Processes of Estimation

When the cooperative study (Reys et al. 1991) on estimation performance and strategies of Japanese students began two years ago, our understanding of the cognitive processes involved in estimation was quite simple: Just round the numbers and compute. Now, we have come to realize the complexity of these cognitive processes. First of all, estimation is a set of integrated processes of many components, such as the knowledge of computational algorithms, ability with mental computation, awareness of different aspects of numbers and operations, ability to recall number facts for "nice" numbers, a feel for accuracy and validity, and so on. This complexity causes students difficulty in estimation and also makes it difficult for us to understand the cognitive processes involved.

Second, estimation has both a procedure-driven aspect and a meaning-driven aspect. The use of the knowledge of a computational algorithm in devising a suitable strategy is one of the keys to successful estimation. However, using one's knowledge of numbers and operations is also a key to success. Some good estimators seem to be able to use both kinds of knowledge. In other words, teaching estimation processes should furnish new opportunities to link conceptual and procedural knowledge in computation.

Finally, there is a simultaneous nature to the use of the cognitive processes involved in estimation. On the one hand, number rounding or mental computation is taking place. On the other hand, one's knowledge of the nature of the numbers and a feeling of accuracy and validity are influencing the rounding and mental computation. To be able to use processes simultaneously seems very difficult for younger students. This might require more time to learn and develop.

IMPLICATIONS FOR INSTRUCTION AND SOME DIRECTIONS FOR FUTURE RESEARCH

Can the Strategies and Processes Used by Good Estimators Be Directly Taught?

The three studies reported here show the cognitive processes and strategies used by good estimators at three school levels. In fifth grade, some good estimators employed taught strategies directly without any adaptation. We also found another type of good estimator who seemed to be more adaptive than the first type. In eighth grade, there were good estimators who used straight rounding and good estimators with more flexible strategies. And in the university freshmen sample, good estimators used combinations of key processes together with appropriate strategies. Even though it was hypothesized that Japanese students would be successful estimators because we teach topics related to estimation, the effects of teaching these topics occurred over a long period of time. It is more likely that becoming a good estimator requires various experiences with numbers and operations. If this supposition is true, estimation topics in the mathematics curriculum should not be placed in a certain grade exclusively. Rather, these topics would be better distributed over all grades.

Who Should Be Called a "Good Estimator"?

This research supports the identification of two types of good Japanese estimators. One type is good at using just one strategy, that is, the taught strategy of rounding for all problems. The other uses several strategies flexibly. Some good older estimators employ several processes and strategies in just one problem. Who, then, should be called a "good estimator"? What are our objectives in teaching estimation? What kind of abilities can we expect students to develop? The practical answer to this question might appear to be, Just develop estimation ability of either type. Various ways of teaching estimation may produce various types of successful estimators. But mathematics educators certainly have an image of a good estimator that is related to the values they implicitly hold. Research on the cognitive processes of estimation will be a major resource in devising appropriate evaluation standards. Therefore, components of "good estimation" must be investigated. In this investigation, identifying the types of estimators who scored high on the estimation test is a first milestone for us.

Relevant Choice of Computational Tools

Students need to learn to choose computational tools depending on the situation, the problem, and their abilities to use these tools. When is an accurate answer really needed in a problem? Can I compute it mentally? Does this take too long when time is limited? These questions should be a part of the important decision making involved in choosing a computational tool. This decision making will be an important ability in situations where several computational alternatives are available. The complementary use of several computational tools will be beneficial to the validity and accuracy of computation. This decision making is crucial in a contemporary society overflowing with data.

People of various abilities must learn to choose which computational tool is relevant in a situation. In addition to research on the cognitive processes associated with using alternative computational methods, the process of deciding which computational alternative should be used must also be targeted for research.

15

Classroom Trials of Teaching Estimation in a Problem-solving Context

Nobuhiko Nohda
Toshiaki Yabe

THE value of teaching estimation and problem solving is commonly acknowledged and is clearly stated both in the current Japanese *Curriculum of Mathematics for the Elementary School* and in the NCTM *Curriculum and Evaluation Standards* (1989). Several theoretical frameworks have been proposed to describe problem solving and the processes involved. For example, Polya (1957) suggested that problem solving consists of four phases: understanding, planning, carrying out the plan, and looking back. A second framework, described by Schoenfeld (1985), describes five steps: analysis, exploration, planning, implementation, and verification. Garofalo and Lester (1985) have constructed a four-step framework to study the cognitive and metacognitive aspects of mathematical performance. Step one, orientation, is the strategic behavior used to assess and understand a problem; step two, organization, is the planning of behavior and the choice of action; step three, execution, is the regulation of behavior to conform to the plan; and step four, verification, is the evaluation of both the decisions made and the outcomes of executed plans.

It seems important to relate three factors—the student, the problem, and the strategies employed—when studying mathematical problem solving. Similar factors have also been identified as important in research on the cognitive processes involved in estimation.

There are divergent viewpoints on the nature of estimation. One view holds that estimation is merely computation, whether mental computation or computation with paper and pencil, abacus (soroban), or calculator. Another view, investigated by the research reported here, is of estimation as a problem-solving activity. From this viewpoint, it is important to study the value of using estimation in a problem-solving context because estimation is useful in understanding the problem, in making a plan, and in evaluating both the problem-solving decisions made and the outcomes of executed plans. From this viewpoint, then, problem solving is a vital context in which to learn and appreciate estimation.

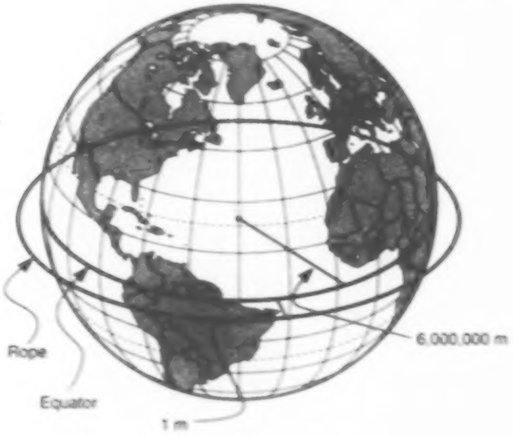
This paper describes a classroom trial of the use of estimation in a problem-solving context and a classroom trial of the use of calculators in the same context. The two trials are compared, and the observed effects from the two classrooms are reported.

THE CLASSROOM TRIALS

The first lesson, conducted near the end of the school year with thirty-five fifth-grade students in the Saitama prefecture, was to teach the use of one estimation strategy in mathematical problem solving. The estimation strategy used was the replacement of given numbers (numbers with more than one significant digit) with simpler numbers to solve a problem. Finding an approximate number is an important part of applied estimation. In this lesson, students use estimation (*a*) to understand a problem and (*b*) to evaluate both the decisions made and the outcome of their executed plans. This classroom lesson, hereafter referred to as Lesson A, was conducted as follows.

The Problem

If we view the earth as a big globe, its radius is about 6,378.136 km. If we position a rope one meter above the equator of the earth, what is the difference between the earth's circumference and the length of that rope?



Procedure

First, the teacher asks the students to predict the difference. The students were asked to estimate and select from the following answers:

1. About 6000 km
2. About 600 km
3. About 1000 m
4. About 6 m
5. We can't guess.

After sharing their estimates, students were asked to solve the problem. At that time, they recalled the formula:

$$\text{circumference} = \text{radius} \times 2 \times 3.14 \quad (\pi = 3.14)$$

But the students had difficulties applying this formula using the exact numbers. Therefore, the teacher suggested using approximate numbers that were simpler. For example:

$$\text{circumference} = 6{,}000{,}000 \times 2 \times 3$$

The students accepted this formula with the simpler numbers and arrived at the following equation:

$$(6{,}000{,}000 + 1) \times 2 \times 3 - 6{,}000{,}000 \times 2 \times 3 = 6$$

After they obtained this solution, the students were asked by the teacher to describe the methods they had used. They gave the following explanations:

1. Use the distributive law:

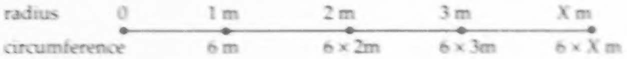
   $$(6{,}000{,}000 + 1) \times 2 \times 3 = 6{,}000{,}000 \times 6 + 6$$

   The answer is the difference is 6 m.

2. Make a table:

radius	1 m	2 m	3 m	X m
circumference	6 m	6 × 2m	6 × 3m	6 × X m

3. Use *senbun zu* (number line):



4. Use a place holder:

   $$(\Delta + 1) \times 2 \times 3 = \Delta \times 6 + 6$$

Finally, the teacher summarized the lesson and asked the students to complete the questionnaire shown in figure 15.1.

1. Circle the number that best describes the way you feel about your regular mathematics lesson.

Like 1 2 3 Dislike

Read each question and circle the number that best matches your impression of this lesson.

Before you solved this problem:

2. Could you understand the problem?

Very well 1 2 3 Couldn't

3. Did you feel that the problem was interesting?

Yes 1 2 3 No

4. Were you confident that you could solve the problem?

Yes 1 2 3 No

5. Could you estimate the answer?

Yes 1 2 3 No

While you were solving the problem:

6. Circle the number that best describes you while you were solving the problem.

a) I could solve the problem in the way I first planned.
b) I changed my original plan to solve the problem.
c) I got stuck and had difficulty.
d) Other

Following the lesson:

7. Can you think of different ways to solve the problem now?

Yes 1 2 3 No

8. Are you satisfied with this lesson?

Yes 1 2 3 No

9. Are you confident of your understanding of the lesson and in the teacher's explanation?

Yes 1 2 3 No

10. Please describe the kind of activity you would like to do in future lessons:

Fig. 15.1. Student questionnaire

In a rural sixth-grade classroom on Kyushu Island, the same problem was posed using numbers with more significant digits and a calculator. For example, the students were given 6,378.136 km as the radius of the earth and used 3.14 as π. It was assumed that calculators would help students calculate these numbers easily. The rest of the lesson, hereafter referred to as Lesson B, was very similar to the previous Lesson A.

Observed Effects

Since these two lessons were only single trials and the classes were quite different, the generality of these effects is limited. Nevertheless, the analysis of students' responses during the two lessons and their answers to the questionnaire at the conclusion of the lessons suggested several effects that might be useful in further research.

1. *Replacing given numbers with simpler numbers helped students to understand the problem.* Sixty-four percent of students using simpler numbers (designated Lesson A) answered yes to Item 2, whereas only 24 percent of the students using a calculator (designated Lesson B) answered yes. This suggests that Lesson A promoted students' understanding of the problem. These two lessons used the same diagram of the earth, and the explanations were very similar. Therefore, the numbers used in the lessons seemed to be the principal factor in students' understanding of the problem. The use of simpler numbers made the structure of the problem more comprehensible because the students could think about the nature of the problem without the complications resulting from the difficult numbers.

2. *By using simpler numbers, students were able to estimate a solution to a problem better than when using the given numbers.* Thirty percent of the students in Lesson A reported they could estimate a solution, whereas only 5 percent of the students in Lesson B said that they were able to make an estimate.

3. *Students who used simpler numbers were able to make reasonable estimates.* During the problem-solving portion of this lesson, the teacher suggested that students use simpler numbers. Figure 15.2 shows students' actual estimates. Six meters is one reasonable estimate, whereas 6000 km would be an order-of-magnitude error. Sixty-one percent of the students in Lesson A were able to make estimates including the digit 6, suggesting their use of $2\pi r$, though almost half of these involved an order-of-magnitude error. However, 68 percent of the students in Lesson B made estimates involving 1, either 1 km or 1000 km. This suggests that in estimating a solution, the students in Lesson A were able to apply the formula and to make an estimate after the teacher's suggestion to use simpler numbers.

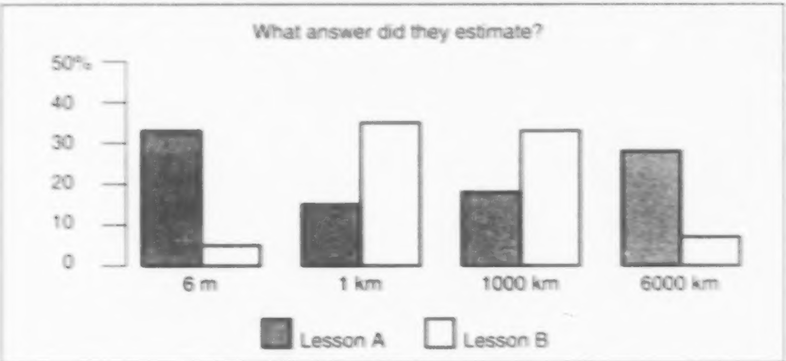


Fig. 15.2. Results of students' estimates

4. *Students who used simpler numbers were able to devise a "nice" plan.* In addition to the effect above on estimated answers, the simpler numbers in Lesson A seemed to enable students to execute their initial plans and to reach answers without any changes in their plan. Figure 15.3 shows students' responses to Item 6. Forty-three percent of the students in Lesson A executed their initial plans without any changes, compared to 11 percent of the students in Lesson B. Figure 15.3 also shows that many more students in Lesson B either changed their plan or were unsuccessful in formulating a plan.

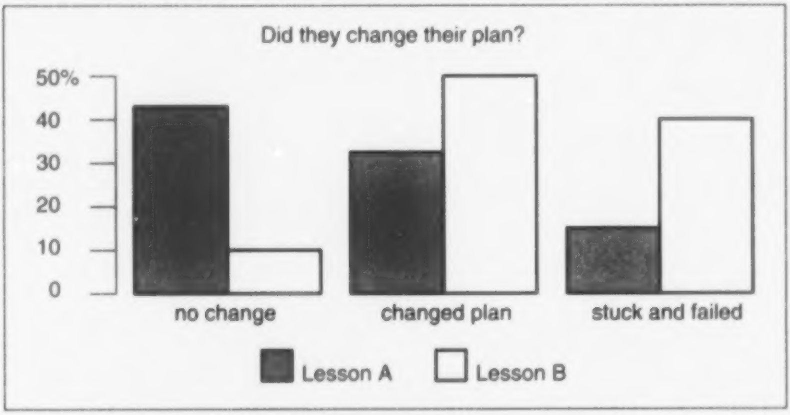


Fig. 15.3. Students' responses regarding changes to their solution plan

More than 60 percent of the students in Lesson A produced a correct solution, whereas a majority of the students in Lesson B were unable to form a correct solution. Thus, students using simpler numbers were better able than those using calculators to formulate and execute a "nice" plan and arrive at a correct solution without needing to alter that plan.

5. *Students using simpler numbers were able to arrive at the synthetic equation that illuminates the structure of the problem.* Figure 15.4 shows the number of students who produced each type of equation. The category "three formulas" means students gave the following solution:

radius × 2 × p = circumference of the earth
(radius + 1) × 2 × p = length of rope
(circumference of the earth) – (length of rope) = difference

Students in the category "synthetic formula" gave the following solution:

(radius + 1) × 2 × p – radius × 2 × p = difference

Thirteen of the thirty-five students in Lesson A produced the synthetic formula, but only two students in Lesson B gave this response. Thirteen of the thirty-three students in Lesson B used three formulas to solve the problem, whereas seven students in Lesson A were in the same category. Students in Lesson A were better able to produce the synthetic equation, in which the structure of the problem is clear. One student in Lesson A gave the following subtracted solution, furnishing further evidence of an understanding of the problem structure:

$$(\text{radius} + 1) \times 2 \times 3 - \text{radius} \times 2 \times 3 = 1 \times 2 \times 3$$

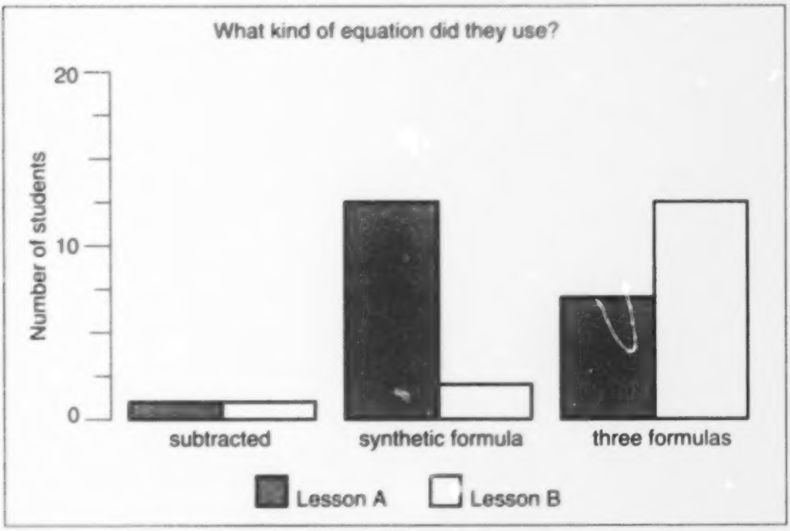


Fig. 15.4. Kinds of equations students used

6. *Students were better able to develop a solution by using simpler numbers than by using the given numbers.* As seen in the previous example, students sometimes develop their initial equations into more sophisticated forms. Ten students in Lesson A actually produced such an activity by, for example, expanding equations and using brackets to unify equations.

7. *Students who used simpler numbers expressed feelings of satisfaction and self-confidence with the lesson.* Figure 15.5 shows percentages for each response on Item 8. Students of both lessons seemed satisfied with the lessons. Sixty-four percent of the students in Lesson A answered "satisfied." Forty-five percent of the students in Lesson B answered "almost satisfied," indicating room for improvement. Yet 12 percent of the students in Lesson A answered "no" on this item. This number is greater than for Lesson B.

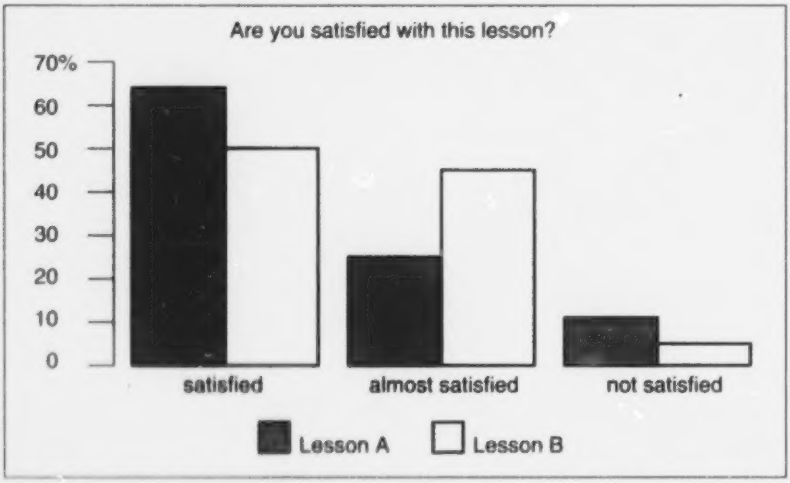


Fig. 15.5. Students' responses to the question "Are you satisfied with this lesson?"

More students (nearly 90 percent) in Lesson A also expressed confidence in their understanding of the lesson and of the teacher's explanation (Item 9) than students in Lesson B (about 60 percent).

GOALS FOR FUTURE RESEARCH

In one of the application lessons described here, a given situation with given numbers was replaced with a simpler situation and simpler numbers. The purposes of the lessons were to develop an appreciation of the merits of using a placeholder (*X* or Δ) and to develop a problem-solving attitude throughout—from an estimation of a problem's solution to the verification of that solution. We conclude with the assertion that the estimation strategy was of value in helping students understand the problem, make a reasonable estimate for the solution, and develop a "nice" solution plan. Besides producing these cognitive benefits, the estimation process seemed to engender in students feelings of satisfaction and self-confidence in their work. Although the extent to which these effects can be generalized is limited at this time, such generalization is a goal for future research.

16

Some Common Directions for Future Research Related to Computational Alternatives

Richard J. Shumway

THE most powerful argument for cross-cultural research is that observed effects in significantly different cultures can be widely generalized and represent fundamental knowledge about human behavior. It is difficult to argue that results found across diverse cultures are attributable to the special circumstances of the specific sample. In fact, such results probably transcend cultural differences and are fundamental results about human behavior, worldwide.

A secondary result for cross-cultural research is that different observed effects can lead to a reexamination of fundamental assumptions and beliefs. Such examination can lead to understood differences, or later to common results through the refinement and control of associated variables.

A PROCESS MODEL FOR COMPUTATIONAL ESTIMATION

In 1980, R. Reys, Bestgen (B. Reys), Rybolt, and Wyatt (1980, 1982) proposed a model for computational estimation consisting of three fundamental processes: *reformulation*, *translation*, and *compensation* (RTC).

The RTC Model

Reformulation: The process of altering numerical data to produce a more mentally manageable form. This process leaves the structure of the problem intact.

Translation: The process of changing the mathematical structure of a problem to a more mentally manageable form

Compensation: Adjustments made to reflect numerical variation that came as a result of translation or reformulation

Until recently, little research has been done using these three processes. However, Sowder and Wheeler (1989) and Case and Sowder (1990) have used them in reporting their work. Furthermore, recent work in both Japan and Mexico has confirmed the existence of these fundamental processes in the estimation strategies of good estimators (Reys et al. 1991; Reys, Reys, and Flores 1991). Consistent with these results, the Japanese research reported in this volume (e.g., Itoh and Mizusawa, chap. 6; Shimizu and Ishida, chap. 14) continues to support the viability of these processes as a conceptual framework for computational estimation strategies.

Although a number of distinct computational estimation strategies have been identified (e.g., front-end, rounding, benchmark, special numbers, clustering, compatible numbers, and flexible rounding), one can argue that there exists no computational estimation strategy yet identified that does not comfortably fit the reformulation/translation/compensation model (the RTC model) for computational estimation processes.

Data from the United States, Japan, and Mexico suggest a remarkable matching of the computational estimation processes of good estimators across several cultures. To confirm or refute this suggestion,

- *the RTC model should be adopted as the model of choice for reporting all research on computational estimation until refinement or change is required.*

Furthermore, the use of these processes when organizing and planning instruction would serve to link research with practice and furnish a viable organizational structure for classroom teachers of computational estimation (Itoh and Mizusawa, chap. 6 in this volume).

IDENTIFICATION OF APPROPRIATE COMPUTATIONAL ALTERNATIVES

The organization of this volume suggests that *written algorithms, mental computation, computational estimation,* and *calculators* represent the principal computational alternatives appropriate for the twenty-first century. Recognizing these categories has allowed researchers to discuss the relative shift in emphasis for these categories in school mathematics (e.g., Reys and Nohda, chap. 1 in this volume). For example, one might propose the estimates shown in table 16.1 of the relative emphasis in different time periods for each of the categories.

Table 16.1.
Percentage of Time Allocated to Various Types of Computation

	Adults	Students		
		Before	Now	Future
Written computation	03	95	85 (80)	05 (10)
Mental computation	02	01	05 (10)	05 (20)
Computational estimation	10	04	05	05 (30)
Calculators	85	00	05	85 (40)

Note: Values are author's hypothetical values; values in parentheses are suggested by Nohda and Shimiza (1991).

The hypothetical figures in table 16.1 suggest a serious mismatch between adult use (column 1 under the heading Adults) and school activity (column 3 under the heading Now). Further, a proposed shift for students in the future (i.e., column 4 under the heading Future) better matches the current use for adults and would be compatible with the proposed view of the calculator as a driving force for computation in the future (Fey 1990). Although one may debate the hypothetical values I have proposed (the parenthetical values indicate one view from Japan), there is unanimous agreement that the curricular and instructional attention now given to different types of computation needs to be reallocated and refocused.

- *What action should be taken to help legitimize different types of computations?*

Table 16.1 would allow one to argue that the calculator's impact should be substantial as compared with the impact of other computational alternatives. Current data on calculator use in both the United States and Japan (see Coburn, chap. 9 in this volume and Senuma, chap. 11) suggest that many do not endorse such a view, nor would some even argue for such pervasive use of calculators (Senuma, chap. 11).

Several researchers in this volume have proposed the widespread use of calculators as a principal motive for giving increased emphasis to computational estimation (Shibata, chap. 2; Itoh and Mizusawa, chap. 6; Schoen, chap. 8). Others would object to the minor role (5 percent) projected for each of mental computation and computational estimation in table 16.1 (for example, see Reys and Nohda, chap. 1). In addition, the proposed use of the word *metacomputation* (Shigematsu, Iwasaki, and Koyama, chap. 3) is important and implies a significant role for mental computation and computational estimation. Yet, it is possible that effective use of calculators requires very minimal and coarse estimation skills. Trafton (chap. 7) proposes a much more central role for mental computation and estimation. If one is to claim that effective use of calculators requires mental computation, computational estimation, and metacomputation, one must clearly demonstrate the relationship among mental computation, computational estimation, metacomputation, and the effective use of calculators (Shibata, chap. 2; Shigematsu, Iwasaki, and Koyama, chap. 3; Itoh and Mizusawa, chap. 6; and Yoshikawa, chap. 5).

- *What are the computational estimation skills needed to use a calculator effectively?*

One might argue that the only computational estimation processes needed to safeguard against wrong answers when using calculators are those that treat order of magnitude, the easiest form of estimation (Schoen, chap. 8). Others make the case for a much more extensive set of processes in order to use calculators effectively.

- *Is the central role for mental computation, computational estimation, and metacomputation one of support only for calculator-based computations or for a deeper role of sense-making in mathematics?*

COMPUTATION FOR THE TWENTY-FIRST CENTURY

A Calculator Base

The calculator will be a substantial computational alternative in the twenty-first century and may provide significant impetus for mental computation, computational estimation, and metacomputation. Moreover, an inexpensive calculator for the twenty-first century is likely to be child-friendly and capable of substantial numeric, graphic, and symbolic computations (Fey 1990, 1992).

Computation for the twenty-first century will surely involve other forms of computation besides numeric, and it would seem natural to move toward a more generalized view of computation (Yoshikawa, chap. 5 in this volume). Thought experiments (Shibata, chap. 2; Wheatley, chap. 10; and Yoshikawa, chap. 5) and students' preference for thinking with "round" numbers rather than using exact numbers with a calculator (Nohda and Yabe, chap. 15) suggest that the knowledge of generalized procedures or the use of "variables" may be critical factors for a future view of computational estimation. Arrays are another form of number effectively treated with calculators (Fey 1990). It seems important to be able to make sense of symbolic and graphic objects. Models for numeric computational estimation could generalize to these other representations. We must begin to find out if they do.

- *Computational estimation must be generalized from numeric objects to symbolic and graphic objects.*

Current curricula use the paper-and-pencil computational algorithms of arithmetic as their foundations. A dramatic shift to a calculator-based curriculum raises serious issues and concerns (Coburn, chap. 9 in this volume; Fey [1990, 1992]; Senuma, chap. 11; Wheatley, chap. 10). We must examine the implications carefully, not for the purposes of blocking a shift to a calculator–based curriculum, but for the purposes of understanding the meaning and implications of such an action.

- *If the calculator is used as the primary tool for computation, what are the expected impacts on learning mathematics?*

We can examine the impacts on currently organized school mathematics topics, but we must go deeper and consider a zero-based curriculum view (National Research Council 1990). With demanding standards and powerful calculators in the hands of all students, we must address exactly what mathematics really is fundamental. What mathematics can we feasibly expect students to encounter and develop?

- *What mathematics must now be considered fundamental in the K–8 curriculum?*

Assuming a rather dramatic change in school mathematics following current national documents arguing for such change (American Association for the Advancement of Science 1989; National Council of Teachers of Mathematics 1989; National Research Council 1989, 1990; Steen 1990), how can we prepare teachers to foster their own growth using powerful calculators as well as foster the growth of their students? As Coburn (chap. 9) observes, none of us has any experience learning mathematics with powerful calculators or examining previously unthought-of concepts and relationships for K–8 mathematics. Teachers will need to be persuaded that they can use powerful calculators and that students will benefit from their use.

- *How can we help teachers use powerful calculators and new curriculum materials?*

Algorithmic Bases for Computational Estimation

Students of algebra (Kieran 1990) and calculus (Dreyfus 1990) appear to use computational algorithms unsuccessfully as a foundation for conceptual thought. In a similar manner, there is substantial evidence that students attempt to use written computational algorithms as a basis for computational estimation (Reys et al. 1991; Shimizu and Ishida, chap. 14 in this volume; Sowder and Wheeler 1989; Sowder, chap. 12; Trafton, chap. 7; Yoshikawa, chap. 5).

Japanese children show less interest in estimation than their United States and Mexican counterparts, perhaps because the Japanese have excellent skills for exact computation. United States and Mexican children seem more willing than the Japanese children to consider computational estimation, perhaps because they are less able to depend on exact mental computation based on algorithms (Reys, Reys, and Flores 1991).

It is important to determine what causes students to use computational algorithms as a foundation for computational estimation, to consider the benefits and disadvantages of such behavior, and to determine appropriate instructional strategies regarding the role of written computational algorithms for computational estimation.

- *What role should paper-and-pencil algorithms play in computational estimation?*

A much more general question is to determine whether or not there is *any* important role for paper-and-pencil algorithms in the twenty-first century.

- *What role, if any, should paper-and-pencil algorithms play in learning mathematics?*

SITUATIONAL ROLES IN COMPUTATIONAL ESTIMATION

Silver (chap. 13 in this volume) makes an important case for the role of situation in mathematical problem solving and suggests that a similar, important role may exist for computational estimation. Several researchers have confirmed the important role situation plays in computational estimation (Schoen, chap. 8; Sowder, chap. 12; Trafton, chap. 7). In most instances, a "realistic" situation or context improves student performance on computational estimation. One potential disadvantage for depending on situations or contexts for the learning of computational estimation processes and strategies is the potential for failure to achieve transfer from one context to another. However, failure to transfer may simply be evidence that the learning goal has not yet been achieved.

- *What are the effects of situation or context on learning, using, and assessing computational estimation strategies?*

Reluctance of Students to Accept Multiple Correct Answers

Data confirm that Japanese students prefer exact answers to "appropriate" estimates (Yoshikawa [1986] and Yoshikawa, chap. 5 in this volume). Silver (chap. 13) describes students who compute exactly and then round off for the requested estimate. Trafton (chap. 7) provides a figure (fig. 7.1) that uses exact answers as a key element. In the spirit of Wheatley (chap. 10), I would claim that *exact answers are a myth*. Only in the most elementary counting situations are exact answers to be preferred.

Young children (and old children) must develop an understanding of the role of approximation theory in the applications of mathematics. Computational estimation seems to be one natural setting for approximation theory. Furthermore, computational estimation is not likely to be valued if exact answers are viewed as possible and desirable.

A caller to a department of mathematics asked for the length of the sides of a square that would have an area of 2,456 square meters. He was given the answer of 49.558 meters and told that this answer was an approximation. The caller asked for more precision, was told 49.5580467735, and was reminded that the answer was still an approximation. The caller asked again that the answer be exact, not knowing that such a request was impossible to meet!

Naïve users of Mathematica, a computer mathematics system, sometimes ask for exact symbolic solutions to cubic or quartic equations, later realizing that the exact formulation, consisting of two to three pages of symbols, was not useful and that a simple set of approximate roots was needed instead. A mature user of mathematics recognizes that exact solutions are rarely possible or desirable. The whole basis for measurement rests on using a finite set of rational numbers to approximate continuous variables. Calculator computations are based on similar assumptions.

The simple purchase of a single item in a store where value-added taxes or sales taxes are in effect demonstrates the folly of exact answers in the most common transaction. Purchase one calculator for $7.95, add 5.75% tax, and just try to pay the cashier exactly $8.407125 for your purchase (7.95 × 1.0575 = 8.407125).

- *Any model for computational estimation must include approximation theory and student beliefs about approximation.*

MENTAL COMPUTATION: SKILL OR PROBLEM SOLVING?

A major difference between Japanese and United States researchers seems to lie with whether mental computation is problem solving (the United States view) or a skill (the Japanese view). This difference has substantial impact on the way one might teach or do research on mental computation. For the Japanese researchers (Shigematsu, Iwasaki, and Koyama, chap. 3 in this volume), there are "best" strategies that one teaches students to use, an idea congruent with the view that mental computation is a skill. For the United States researchers (Reys and Barger, chap. 4), one needs a variety of strategies available. A critical element is strategy selection, an idea congruent with the view that mental computation is problem solving.

It appears to me that both camps, the robots (skills approach) and the accidental learners (problem-solving approach), have something to learn from each other. Surely there are skills involved with mental computation. Just as computational estimation benefits from the RTC model, there must be elements of the RTC model that are used for mental computations. One must be alert to the problems of decomposition and decontextualization that could accompany a skill-learning approach. Also, one must guard against the lack of specificity and direction for students that may result from a problem-solving approach.

- *What are the skills, concepts, and problem-solving behaviors necessary for effective mental computation?*

One possible way out of the apparent disagreement between Japanese and United States researchers could be to decide that mental computation is quick computation done mentally that does not involve RTC processes. Activities that involve RTC processes yet yield exact answers are simply computational estimates with a high-level error compensation.

METACOMPUTATION

Shigematsu, Iwasaki, and Koyama (chap. 3) have suggested that a new term, *metacomputation,* be used in discussions of computational alternatives for the twenty-first century. To illustrate, they propose mental computation as a metacomputation to guide the use of calculators. Perhaps we can view metacomputation as involving processes and strategies employed to guide computational choices.

It would seem that this idea of metacomputation would accommodate issues of thought experiments, number sense, and mathematical sense making, as well as the four areas chosen for the organization of this volume—written algorithms, mental computation, computational estimation, and calculators.

- *Define "metacomputation" and determine whether or not it is a useful concept for describing and thinking about issues of computation for the twenty-first century.*

COMPUTATIONAL ALTERNATIVE IMPACTS FOR TWENTY-FIRST CENTURY MATHEMATICS

Curriculum

Reys and Barger (chap. 4) and Trafton (chap. 7) make particularly strong cases for a central role for mental computation and computational estimation in school mathematics. Mental computation and computational estimation are advocated as vehicles for promoting thinking, conjecturing, and generalizing on the basis of conceptual understandings. Computational estimation is an important part of a more general activity of making sense out of mathematics.

- *In what way do mental computation and computational estimation promote broader understanding of other mathematical ideas?*

Those believing in such broad benefits of mental computation and computational estimation must show that there really are important connections to be made and that computational estimation makes a contribution that is not possible through more direct treatment of the same connected topics.

Assessment

What are the implications for assessment resulting from the widespread use of powerful calculators, creative measurement of mental computation, computational estimation, and metacomputation? Major work in innovative methods of assessment must accompany all the proposed research activities.

- *New and varied assessment procedures must accompany all research efforts associated with computation for the twenty-first century.*

SUMMARY

The most critical objectives for the future of research on computational alternatives are the following:

1. Use or refute the RTC model.
2. Plan for major changes in emphasis on various computational alternatives.
3. Identify fundamental principles. For example:
   a) Define and use metacomputation.
   b) Generalize computational estimation to symbolic and graphic objects.
   c) Identify fundamental mathematical concepts and relationships.
   d) Incorporate approximation theory in computational estimation.
4. Involve teachers in all aspects of the research process.
5. Continue cross-cultural research activities.

It is clear that marked progress has been made on computational alternatives by the very production of this volume. We have provided ourselves with a playing field, some rules of the game, and some key players. The next step is to throw out the first pitch.

English References

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